GLOBAL EXISTENCE OF WEAK SOLUTIONS TO THE FENE DUMBBELL MODEL OF POLYMERIC FLOWS

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Abstract
Systems coupling fluids and polymers are of great interest in many branches of sciences. One of the models to describe them is the FENE (Finite Extensible Nonlinear Elastic) dumbbell model. We prove global existence of weak solutions to the FENE dumbbell model of polymeric flows for a very general class of potentials. The main problem is the passage to the limit in a nonlinear term that has no obvious compactness properties. The proof uses many weak convergence techniques. In particular it is based on the control of the propagation of strong convergence of some well chosen quantity by studying a transport equation for its defect measure.

1. INTRODUCTION

Systems coupling fluids and polymers are of great interest in many branches of applied physics, chemistry and biology. They are of course used in many industrial and medical applications such as food processing, blood flows... Although a polymer molecule may be a very complicated object, there are simple theories to model it. One of these models is the FENE (Finite Extensible Nonlinear Elastic) dumbbell model. In this model, a polymer is idealized as an "elastic dumbbell" consisting of two "beads" joined by a spring which can be represented by a vector R (see Bird, Curtis, Amstrong and Hassager [7, 8], Doi and Edwards [18] for some physical introduction to the model and Ottinger [55] for a more mathematical treatment (in particular the stochastic point of view) of it and Owens and Phillips [57] for the computational aspect). In the FENE model (1), the polymer elongation R cannot exceed a limit R_0. This yields some nice mathematical problems near the boundary, namely when $|R|$ approaches R_0. At the level of the polymeric liquid, we get a system coupling the Navier-Stokes equation for the fluid velocity with a Fokker-Planck equation describing the evolution of the polymer density. This density depends on t, x and R. The coupling comes from an extra stress term in the fluid equation due to the microscopic effect of the polymers. This is the micro-macro interaction. There is also a drift term in the Fokker-Planck equation that depends on the spatial gradient of the velocity. This is a macro-micro term. The coupling satisfies the fact that the free-energy dissipates which is important from the physical point of view. Mathematically, this is also important to get uniform bounds and hence prove global existence of weak solutions.

The system obtained attempts to describe the behavior of this complex mixture of polymers and fluid, and as such, it presents numerous challenges, simultaneously at the level of their derivation [15], the level of their numerical simulation [57, 34], the level of their physical properties (rheology) and that of their mathematical treatment (see references below). In this paper we concentrate on the mathematical treatment and more precisely the global existence of weak solutions to the FENE dumbbell model (1). These solutions are the generalization

of the Leray weak solutions [43, 42] of the incompressible Navier-Stokes system to the FENE model.

An approximate closure of the linear Fokker-Planck equation reduces the description to a closed viscoelastic equation for the added stresses themselves. This leads to well-known non-Newtonian fluid models such as the Oldroyd B model or the FENE-P model (see for instance [19, 15]). These models have been studied extensively. Guillopé and Saut [26, 27] proved the existence of local strong solutions, Fernández-Cara, Guillén and Ortega [22], [21] and [23] proved local well posedness in Sobolev spaces. In Chemin and Masmoudi [9] local and global well-posedness in critical Besov spaces was given. For global existence of weak solutions, we refer to Lions and Masmoudi [48]. We also mention Lin, Liu and Zhang [45] where a formulation based on the deformation tensor is used to study the Oldroyd-B model. Global existence for small data was also proved in [41, 39].

At the micro-macro level, there are also several works. Indeed, from the mathematical point of view, the FENE model and some simplifications of it were studied by several authors. In particular Renardy [58] proved the local existence in Sobolev space where the potential $\mathcal{U}$ is given by $\mathcal{U}(R) = (1-|R|^2)^{1-\sigma}$ for some $\sigma > 1$. W. E, Li and Zhang [20] proved local existence when R is taken in the whole space and under some growth condition on the potential. Also, Jourdain, Lelievre and Le Bris [33] proved local existence in the case $b = 2k > 6$ for a Couette flow by solving a stochastic differential equation (see also [31] for the use of entropy inequality methods to prove exponential convergence to equilibrium). Zhang and Zhang [61] proved local well-posedness for the FENE model when $b > 76$. Local well-posedness was also proved in [51] when $b = 2k > 0$ (see also [36]). One of the main ingredients of [51] is the use of Hardy type inequalities to control the extra stress tensor by the H^1 norm in R which comes from the diffusion in R. In particular no regularity in R is necessary for the initial data. Moreover, Lin, Liu and Zhang [46] proved global existence near equilibrium under some restrictions on the potential (see also the related work [39]). Recently many other works dealt with different aspect of the system. In particular the problem in a thin film was considered in [11], the problem of the long time behavior was considered in [60, 30, 1], the problem of global existence in smooth spaces in 2D for some simplified models (when there is a bound on τ in L^∞) was considered in [13, 47, 14, 54], the problem of non-blow up criterion was considered in [40], the problem of stationary solution was considered in [11, 10], the study of the boundary condition at ∂B was considered in [28, 50].

More related to this paper, the construction of global weak solutions for simplified models was considered in [3, 4, 5, 62, 60, 6] in the case the system is regularized by some diffusion in the space variable or by a microscopic cut-off. The case of the co-rotational model was considered in [49]. The co-rotational model preserves some of the compactness difficulties of the full model. It allows to get more integrability on the ψ which makes the compactness analysis much simpler.

We end this introduction by mentioning other micro-macro models. Indeed, a principle based on an energy dissipation balance was proposed in [12], where the regularity of nonlinear Fokker-Planck systems coupled with Stokes equations in 3D was also proved. In particular the Doi model (or Rigid model) was considered in [56] where the linear Fokker-Planck system is coupled with a stationary Stokes equations. The nonlinear Fokker-Planck equation driven by a time averaged Navier-Stokes system in 2D was studied in [13] (see also [14]). Recently, there were many review papers dealing with different mathematical aspects of these models [59, 44, 38]. In particular we refer to [38] for an exhaustive list of references dealing with the numerical point of view.

1.1. The FENE model.

A macro-molecule is idealized as an "elastic dumbbell" consisting of two "beads" joined by a spring which can be modeled by a vector R (see [8]). Before writing our main system (1), let us discuss the main physical assumptions that lead to it:

- The polymers are described by their density at each time t, position x and elongation R. This is a *kinetic description* of the polymers.
- The inertia of the polymers is neglected and hence the sum of the forces applied on each polymer vanishes. We refer to [16] where inertia is taken into account. Moreover, the limit m goes to zero it studied where m is the mass of the beads.
- The polymer solution is supposed to be dilute and hence the interaction between different polymers is neglected. This is why we get a linear Fokker-Planck equation. Let us also mention that there are models for polymer melts such as the reptation model (see for instance [55]).
- The polymer is described by one vector R in $B(0, R_0)$. Let us mention that there are models where each polymers is described by one vector R such that $|R| = 1$ (the rigid case, see [14]) or by K vectors R_i, $1 \leq i \leq K$ (see [6]). Usually the difference between these models comes from the length of the polymers as well as their electric properties.
- In the Fokker-Planck equation an upper-convected derivative is used. This is can be seen as the most physical one. Other used derivatives are the lower-convected and the co-rotational ones (see [7, 8]). The co-rotational one has the mathematical advantage that one has better a priori estimates (see [49]).
- We neglect the diffusion in x in the Fokker-Planck equation. Indeed, this diffusion is much smaller than the diffusion in R. Actually, it makes the mathematical problem much simpler.

Under these assumptions, the micro-macro approach consists in writing a coupled multi-scale system :

$$
\left\{
\begin{array}{l}
\partial_t u + (u \cdot \nabla)u - \nu \Delta u + \nabla p = \mathrm{div}\tau, \quad \mathrm{div} u = 0, \\
\\
\partial_t \psi + u.\nabla \psi = \mathrm{div}_R \Big[- \nabla u \, R\psi + \beta \nabla \psi + \nabla \mathcal{U} \psi \Big] \\
\\
\tau_{ij} = \int_B (R_i \otimes \nabla_j \mathcal{U}) \psi(t, x, R) dR \qquad (\nabla \mathcal{U} \psi + \beta \nabla \psi).n = 0 \text{ on } \partial B(0, R_0).
\end{array}
\right. \tag{1}
$$

In (1), $\psi(t, x, R)$ denotes the distribution function for the internal configuration and $F(R) = \nabla_R \mathcal{U}$ is the spring force which derives from a potential $\mathcal{U}$ and $\mathcal{U}(R) = -k\mathrm{log}(1 - |R|^2/|R_0|^2)$ for some constant $k > 0$. Besides, β is related to the temperature of the system and $\nu > 0$ is the viscosity of the fluid. In the sequel, we will take $\beta = 1$.

Here, R is in a bounded ball $B(0, R_0)$ of radius R_0 which means that the extensibility of the polymers is finite and $x \in \Omega$ where Ω is a bounded domain of $\mathbb{R}^D$ where $D \geq 2$ or $\Omega = \mathbb{T}^D$ or $\Omega = \mathbb{R}^D$. In the case Ω has a boundary, we add the Dirichlet boundary condition $u = 0$ on $\partial\Omega$. We have also to add a boundary condition to insure the conservation of ψ, namely $(-\nabla u R\psi + \nabla \mathcal{U}\psi + \beta \nabla \psi).n = 0$ on $\partial B(0, R_0)$. The boundary condition on $\partial B(0, R_0)$ insures the conservation of the polymer density and should be understood in the weak sense, namely for any function $g(R) \in C^1(B)$, we have

$$
\partial_t \int_B g\psi dR + u.\nabla_x \int_B g\psi dR = - \int_B \nabla_R g \Big[- \nabla u \, R \psi + \beta \nabla \psi + \nabla \mathcal{U} \psi \Big] dR. \tag{2}
$$

Notice in particular that it implies that $\psi = 0$ on $\partial B(0, R_0)$ and that if initially $\int \psi(t = 0, x, R) dR = 1$, then for all t and x, we have $\int \psi(t, x, R) dR = 1$. We will see later an other way of understanding this singular boundary condition.

When doing numerical simulation on the FENE model, it is usually better to think of the distribution function ψ as the density of a random variable R which solves (see [55])

$$
dR + u.\nabla R dt = (\nabla u R - \nabla_R \mathcal{U}(R))dt + \sqrt{2} dW_t \tag{3}
$$

where the stochastic process W_t is the standard Brownian motion in $\mathbb{R}^N$ and the additional stress tensor is given by the following expectation $\tau = \mathbb{E}(R_i \otimes \nabla_j \mathcal{U})$. Of course, we may need

to add a boundary condition for (3) if R reaches the boundary of B. This is done by requiring that R stays in $\overline{B}$ (see [32]). Using this stochastic formulation has the advantage of replacing the second equation of (2.1) which has $2D+1$ variables by (3). Of course one has to solve (3) several times to get the expectation τ which is the only information needed in the fluid equation. This strategy was used for instance by Keunings [35] (see also [24]) and by Öttinger [55] (see also [25]).

In the sequel, we will only deal with the FENE model and we will take $\beta = 1$ and $R_0 = 1$.

2. Statement of the results

This paper is devoted to the proof of global existence of free-energy weak solutions to the FENE model. The main difficulty of the construction is the passage to the limit in an approximate system in the nonlinear term $\nabla u^n \psi^n$. Indeed, we only have a uniform bound on ∇u^n in $L^2((0,T)\times\Omega)$ and ψ^n in $L^\infty((0,T)\times\Omega; L^1(B))$ for all $T > 0$ and so assuming that u^n and ψ^n converge weakly to u and ψ, it is not clear how to deduce that $\nabla u^n \psi^n$ converges weakly to $\nabla u \psi$.

Before mentioning our main result, let us recall that the construction of global weak solutions to simplified models was considered in [4, 5, 60, 49, 62]. In particular in [4] a diffusion in the space variable in the ψ equation is added. Mathematically this yields a bound on $\nabla_x \sqrt{\psi}$ in $L^2((0,T)\times\Omega\times B)$ and hence one can easily pass to the limit in the product $\nabla u^n \psi^n$ using the Lions-Aubin lemma. This extra diffusion term is physically justifiable but it is much smaller than the diffusion in the R variable and this is why we did not include it here. Recently, Barrett and Suli [6] extended their results to the case of bead-spring chain models where each polymer is described by K springs R^i, $1 \le i \le K$ again with diffusion in the x variable. Also, in [49], the co-rotational model was considered. It allowed us to get more a priori estimates on ψ^n, namely one can get that ψ^n is in all L^p spaces. An argument based on propagation of compactness similar to the one used in [48] allowed us to conclude.

Here, we consider the more physical model (1). The system (1) has to be complemented with an initial data $u(t=0) = u_0$ and $\psi(t=0) = \psi_0$.

Notice that $(u = 0, \psi_\infty)$ where ψ_∞

$$\psi_\infty(R) = \frac{e^{-\mathcal{U}(R)}}{\int_B e^{-\mathcal{U}(R')} dR'} \tag{4}$$

defines a stationary solution of (1). To state our result, we first impose some conditions on the initial data. We take $u_0(x) \in L^2(\Omega)$, $\mathrm{div}(u_0) = 0$ and $\psi_0(x,R) \ge 0$ such that $\rho_0(x) = \int \psi_0 dR \in L^\infty(\Omega)$. Here $\rho_0(x)$ is the initial density of polymers at the position x. We also assume the following entropy bound : $\frac{\psi_0}{\rho_0\psi_\infty} \in L\log L(\Omega\times B, dx\rho_0(x)\psi_\infty dR))$, namely

$$\|\frac{\psi_0}{\rho_0\psi_\infty}\|_{L\log L(\Omega\times B,\rho_0(x)\psi_\infty dRdx)} = \int\int_{\Omega\times B} (\frac{\psi_0}{\rho_0\psi_\infty}\log\frac{\psi_0}{\rho_0\psi_\infty} - \frac{\psi_0}{\rho_0\psi_\infty} + 1)\rho_0(x)\psi_\infty dRdx < \infty. \tag{5}$$

Finally, we also assume the following $L_x^{1/2} L\log^2 L$ bound, that we will call "$\log^2$" bound:

$$\int_\Omega \frac{\int_B \psi_0 \log^2 \frac{\psi_0}{\rho_0\psi_\infty}}{1 + \left[\int_B \psi_0 \log^2 \frac{\psi_0}{\rho_0\psi_\infty}\right]^{1/2}} dx < \infty. \tag{6}$$

Notice that interpolating (6) with the L^∞ bound on ρ_0, we can deduce the $L\log L$ bound (5).

Theorem 2.1. *Take a divergence free field $u_0(x) \in L^2(\Omega)$ and $\psi_0(x,R) \ge 0$ such that $\rho_0(x) = \int \psi_0 dR \in L^\infty(\Omega)$ and (5) and (6) hold. Then, (1) has a global weak solution (u,ψ) such that $u \in L^\infty(\mathbb{R}_+; L^2) \cap L^2(\mathbb{R}_+; \dot H^1)$, $\frac{\psi}{\rho\psi_\infty} \in L^\infty(\mathbb{R}_+; L\log L(\Omega\times B, dx\rho(x)\psi_\infty dR)))$ where*

$\rho(x) = \int_B \psi dR$ and $\sqrt{\frac{\psi}{\psi_\infty}} \in L^2(\mathbb{R}_+; L^2(\Omega; \dot{H}^1_R(\psi_\infty dR)))$ and (32) holds with an inequality $\leq$ instead of the equality and (42) holds (with Ω replaced by any compact K of Ω in the whole space case).

Remark 2.2. *1) Of course u and ψ have also some time regularity in some negative Sobolev spaces in x and R. This allows to give a sense to the initial data (see [48] for more details).*

2) By $f \in L\log L(\Omega \times B, dx\rho(x)\psi_\infty dR)$ we mean that $\int\int_{\Omega\times B}(f\log f - f + 1)\rho(x)\psi_\infty dR < \infty$. Notice that (5) does not really define a norm. One can of course define a norm using Orlicz spaces. However, we do not need to do it here.

3) If the domain Ω has finite measure (bounded domain or torus) then, the extra bound (6) reduces to $\int_\Omega \left[\int_B \psi_0 \log^2 \frac{\psi_0}{\rho_0\psi_\infty}\right]^{1/2} dx < \infty$. This extra bound on the initial data allows us to prove the extra bound (42) on the solution. This is useful to get some sort of equi-integrability of the extra stress tensor. Of course this is a very mild extra assumption, but it would be nice to see if one can prove the same result without it. Moreover, due to the local character of the weak compactness proof, the assumption (42) can be weakened by assuming the bound to hold locally in space, namely $\int_K \left[\int_B \psi_0 \log^2 \frac{\psi_0}{\rho_0\psi_\infty}\right]^{1/2} dx < \infty$ for any compact set K of Ω.

4) For the simplicity of the presentation, the proof will be given in the case $\rho_0(x)$ is constant equal 1 and Ω has finite measure. We will also indicate the necessary changes to be done in the general case.

The paper is organized as follows. In the next section, we give some preliminaries where we prove some Hardy type inequalities. In section 4, we derive some a priori estimates for the full model (1). In particular we recall the free energy estimate as well as a new "log^2" a priori estimate which is useful in controlling the transport of the defect measures. In section 5, we prove the main theorem 2.1. As is classical when proving global existence of weak solutions, the only none trivial part is the proof of the weak compactness of a sequence of global solutions satisfying the a priori estimates and we will only detail this part of the proof. In section 6, we present one way of approximating the system. In section 7 we present some concluding remarks and some open problems.

3. Preliminaries

3.1. **Hardy type inequalities.** The dissipation term in the free energy estimate (32) measures the distance between ψ and the equilibrium ψ_∞. We would like to use that bound to control the extra stress tensor in L^2. This will be done using the following Hardy [29] type inequality.

Lemma 3.1. *If $k > 1$, then we have*

(7)
$$\int_0^1 \frac{\psi}{x^2} \leq C\int_0^1 x^k \left|\left(\sqrt{\frac{\psi}{x^k}}\right)'\right|^2 + \psi.$$

For $k > 0$, we have

(8)
$$\left(\int_0^1 \frac{\psi}{x}\right)^2 \leq C\left(\int_0^1 \psi\right)\left(\int_0^1 x^k|\left(\sqrt{\frac{\psi}{x^k}}\right)'|^2 + \psi\right).$$

For $-1 \leq \beta < k \leq 1$, we have

(9)
$$\left(\int_0^1 \frac{\psi}{x^{1+\beta}}\right) \leq C\left(\int_0^1 \psi\right)^{\frac{1-\beta}{2}} \left(\int_0^1 x^k|\left(\sqrt{\frac{\psi}{x^k}}\right)'|^2 + \psi\right)^{\frac{1+\beta}{2}}$$ *and more generally for all $\gamma \geq 0$*

(10)
$$\left(\int_0^1 \frac{\psi \log^\gamma\left(C+\frac{\psi}{x^k}\right)}{x^{1+\beta}}\right) \leq C\left(\int_0^1 \psi \log^{\frac{2\gamma}{1-\beta}}\left(C+\frac{\psi}{x^k}\right)\right)^{\frac{1-\beta}{2}} \left(\int_0^1 x^k |\left(\sqrt{\frac{\psi}{x^k}}\right)'|^2+\psi\right)^{\frac{1+\beta}{2}}.$$

Remark 3.2. *Before giving the proof, let us mention that this lemma should be compared to the results of section 3.2 of* [51]. *In particular Proposition 3.1 was used to control the extra stress tensor. However, the main difference is that the results of section 3.2 of* [51] *are done in an L^2 frame work since we were dealing with strong solutions there, however the results of lemma 3.1 are in an L^1 frame work since we only have a control on the free energy and its dissipation.*

Inequality (7) for $k>1$ is just Hardy inequality. Notice that there is no requirement on the boundary data since $k>1$. To prove it, we make the change of variable $y=x^{1-k}$ and $h(y)=\sqrt{\frac{\psi(x)}{x^k}}$. Hence, to prove (7), it is enough to prove that

$$\int_1^\infty \frac{h^2}{y^2}dy \leq C\int_1^\infty h'(y)^2+\frac{h^2}{y^{2\alpha}}dy \tag{11}$$

where $\alpha=\frac{k}{k-1}>1$. To prove (11), we integrate by parts in

$$\int_1^A \frac{h\,h'}{y}dy=\int_1^A \frac{h^2}{2y^2}dy+\frac{h(A)^2}{2}-\frac{h(1)^2}{2} \tag{12}$$

for each $A>1$. The left hand side is bounded by $C(\int_1^A \frac{h^2}{y^2}dy)^{1/2}\ (\int_1^A h'(y)^2dy)^{1/2}$. To bound, $h(1)^2$ by the right hand side of (11), we use that $h(y)\leq C\sqrt{y}$ since $\int_1^\infty h'(y)^2dy<\infty$ hence, $\frac{h^2}{y^\alpha}$ goes to zero when y goes to infinity. This yields that

$$h^2(1)=-\int_1^\infty \left(\frac{h^2}{y^\alpha}\right)'dy=-\int_1^\infty 2\frac{h}{y^\alpha}h'-\alpha\frac{h^2}{y^{\alpha+1}}dy \tag{13}$$

which is controlled by the right hand side of (11) using Cauchy-Schwarz and the fact that $\alpha>1$. Letting A go to infinity, we get the result.

The proof of (8) when $k>1$ follows by interpolation.

In the case $0<k\leq 1$, (7) only holds if we add a vanishing boundary condition at $x=0$. However, we can still prove that (8) holds without any extra condition. Indeed, making the change of variables $y=x^{1-k}$ (when $k<1$) and denoting $h(y)=\sqrt{\frac{\psi(x)}{x^k}}$, we see that (8) is equivalent to

$$\left(\int_0^1 y^{\alpha-1}h^2dy\right)^2 \leq C\left(\int_0^1 y^{2\alpha}h^2\,dy\right)\ \left(\int_0^1 h'(y)^2+y^{2\alpha}h^2\right) \tag{14}$$

where $\alpha=\frac{k}{1-k}$. To prove (14), we integrate by parts in the following integral :

$$\int_0^1 y^\alpha h\ h'dy=-\frac{\alpha}{2}\int_0^1 y^{\alpha-1}h^2+\frac{h^2(1)}{2}. \tag{15}$$

and notice that the left hand side is bounded by $\left(\int_0^1 y^{2\alpha}h^2\ \int_0^1 h'(y)^2\right)^{1/2}$ using Cauchy-Schwarz inequality.

Moreover, we have

$$
\begin{aligned}
h(1)^2 = \int_0^1 (y^{2\alpha+1}h^2)'dy &= \int_0^1 y^{2\alpha+1} h\, h' + (2\alpha+1) y^{2\alpha} h^2 \\
&\le C \left(\int_0^1 h'(y)^2 + y^{2\alpha} h^2 \int_0^1 y^{2\alpha} h^2 \right)^{1/2}. \qquad (16)
\end{aligned}
$$

Hence, (14) follows.

When $k = 1$, we make the change of variable $y = -\log x$ and hence (8) is equivalent to

$$
\left(\int_0^\infty e^{-y} h^2 dy \right)^2 \le C \left(\int_0^\infty e^{-2y} h^2\, dy \right) \left(\int_0^\infty h'(y)^2 + e^{-2y} h^2 \right) \qquad (17)
$$

and the proof of (17) can be done in a similar way as that of (14).

To prove (9), we first notice that if $-1 \le \beta \le 0$, then the inequality can be easily deduced from (8) by interpolation. When $\beta > 0$, (9) is equivalent (in the case $k < 1$) to

$$
\left(\int_0^1 y^{\alpha_\beta - 1} h^2 dy \right)^2 \le C \left(\int_0^1 y^{2\alpha} h^2\, dy \right)^{\frac{1-\beta}{2}} \left(\int_0^1 h'(y)^2 + y^{2\alpha} h^2 \right)^{\frac{1+\beta}{2}} \qquad (18)
$$

where $\alpha_\beta = \frac{k-\beta}{1-k}$ and $\alpha = \frac{k}{1-k}$. Applying (14) with α replaced by α_β, we get

$$
\left(\int_0^1 y^{\alpha_\beta - 1} h^2 dy \right) \le C \left(\int_0^1 y^{2\alpha_\beta} h^2\, dy \right)^{1/2} \left(\int_0^1 h'(y)^2 + y^{2\alpha} h^2 \right)^{1/2} \qquad (19)
$$

Notice that we kept α in the last term instead of putting α_β. Indeed, the last integral comes from the estimate of $h^2(1)$ and we can keep $\alpha = \frac{k}{1-k}$ in (16). Now, we can apply (19) replacing $\alpha_\beta - 1$ by $2\alpha_\beta$ and we get

$$
\left(\int_0^1 y^{2\alpha_\beta} h^2 dy \right) \le C \left(\int_0^1 y^{2(2\alpha_\beta+1)} h^2\, dy \right)^{1/2} \left(\int_0^1 h'(y)^2 + y^{2\alpha} h^2 \right)^{1/2} \qquad (20)
$$

We can iterate this, replacing $\alpha - 1$ by $2\alpha_\beta$, $2(2\alpha_\beta + 1)$, ... in (19) till we get an index greater than $2\alpha = 2\frac{k}{1-k}$. Interpolating with the last inequality, yields (9).

In the case $k = 1$, (9) is equivalent to

$$
\left(\int_0^\infty e^{-(1-\beta)y} h^2 dy \right) \le C \left(\int_0^\infty e^{-2y} h^2\, dy \right)^{\frac{1-\beta}{2}} \left(\int_0^\infty h'(y)^2 + e^{-2y} h^2\; dy \right)^{\frac{1+\beta}{2}} \qquad (21)
$$

The proof of (21) is similar and is left to the reader.

For the proof of (10), we use that it is equivalent (in the case $k < 1$) to

(22)

$$
\left(\int_0^1 y^{\alpha_\beta - 1} h^2 \log^\gamma(h^2) dy \right)^2 \le C \left(\int_0^1 y^{2\alpha} h^2 \log^{\frac{2\gamma}{1-\beta}}(h^2)\, dy \right)^{\frac{1-\beta}{2}} \left(\int_0^1 h'(y)^2 + y^{2\alpha} h^2 \right)^{\frac{1+\beta}{2}}.
$$

Again, one can prove (22) in the case $\beta = 0$ by an integration by parts similar to the one used in (14). The case where $-1 \le \beta \le 0$ can be deduced by interpolation from the case $\beta = 0$ and the case $0 < \beta < k$ can be deduced by a bootstrap argument as the one used in the proof of (9).

3.2. **Control of the stress tensor.** We recall that $\psi_\infty(R) = \frac{e^{-\mathcal{U}(R)}}{\int_B e^{-\mathcal{U}(R')} dR'} = (1-|R|^2)^{k/\beta} / \int_B (1-|R'|^2)^{k/\beta}\, dR'$ and since $\beta = 1$, $\psi_\infty(R)$ behaves like $(1-|R|)^k$ when $|R|$ goes to 1. In particular we will apply lemma 3.1 with $x = 1 - |R|$.

Using the inequality (8) in the radial variable with $x = 1 - |R|$, we get

Corollary 3.3. *There exists a constant C such that we have the following bound*

$$|\tau(\psi)|^2 \leq (\int_B \psi dR) \int_B \left|\nabla_R \sqrt{\frac{\psi}{\psi_\infty}}\right|^2 \psi_\infty \, dR \tag{23}$$

This Corollary can be seen as the L^1 version of Proposition 3.1 of [51]. It will allow us to control the extra stress tensor by the free energy dissipation.

3.3. **Weighted Sobolev inequality.** In subsection (5.1), we have to prove the equi-integrability of N_2^n. This will require the control of some higher L^p space of $\sqrt{\frac{\psi}{\psi_\infty}}$. We have the following proposition

Proposition 3.4. *There exists $p > 2$ and a constant C such that we have the following bound*

$$\left(\int_B \left|\sqrt{\frac{\psi}{\psi_\infty}}\right|^p \psi_\infty\right)^{1/p} \leq \left(\int_B \left|\nabla_R \sqrt{\frac{\psi}{\psi_\infty}}\right|^2 \psi_\infty + \psi \, dR\right)^{1/2}. \tag{24}$$

For the proof we first notice that the only difficulty comes from the weight and hence we can restrict to the region where $|R| > \frac{1}{2}$. We also use some spherical coordinates, namely $R = (1-x)\omega$ where $\omega \in \mathbb{S}^{D-1}$ and $0 < x < \frac{1}{2}$. The square of the right hand side of (24) can be written as the sum of a radial part and an angular part :

$$\int_{\mathbb{S}^{D-1}} \left(\int_0^{1/2} \left[\left|\partial_x \sqrt{\frac{\psi}{\psi_\infty}}\right|^2 + |\sqrt{\frac{\psi}{\psi_\infty}}|^2\right] x^k \, dx\right) d\omega. \tag{25}$$

$$\int_0^{1/2} \int_{\mathbb{S}^{D-1}} \left(\int_{\mathbb{S}^{D-1}} \left[\left|\partial_\omega \sqrt{\frac{\psi}{\psi_\infty}}\right|^2 + |\sqrt{\frac{\psi}{\psi_\infty}}|^2\right] d\omega\right) x^k \, dx. \tag{26}$$

We recall the following 1D weighted $L^p - L^q$ Hardy inequality (one can also call it weighted Sobolev inequality)

$$\left(\int_0^{1/2} |F(x)|^q x^k \, dx\right)^{1/q} \leq C \left(\int_0^{1/2} |F'(x)|^2 x^k \, dx\right)^{1/2}. \tag{27}$$

This inequality can be easily deduced from Theorem 6 of [37], taking $u(x) = v(x) = x^k$ for any $q < \infty$ if $k \leq 1$ and for $q \leq \frac{2(k+1)}{k-1}$ if $k > 1$. Indeed, Theorem 6 of [37] stated that (27) holds for any F, with $F(\frac{1}{2}) = 0$ if

$$\sup_{0<r<\frac{1}{2}} (\int_0^r x^k dx)^{1/q} (\int_r^{\frac{1}{2}} (x^k)^{-1} dx)^{1/2} < \infty.$$

Hence, we get a control of $\sqrt{\frac{\psi}{\psi_\infty}}$ in the space $L^2(\mathbb{S}^{D-1}; L^q((0,\frac{1}{2}), x^k dx))$ using the radial part of the norm (25).

On the other hand we can use the classical Sobolev inequality in $D-1$ dimension to control $\sqrt{\frac{\psi}{\psi_\infty}}$ in the space $L^2_x((0,\frac{1}{2}); L^s(\mathbb{S}^{D-1}), x^k dx)$ where $s = \frac{2(D-1)}{(D-1)-2}$ if $D > 3$, $s < \infty$ if $D = 3$ and $s \leq \infty$ if $D = 2$. Interpolating between the two spaces $L^2_\omega L^q_x$ and $L^2_x L^s_\omega$, we deduce the existence of some $p > 2$ such that (24) holds.

3.4. **Young measures and Chacon limit.** We recall here two important weak convergence objects used in this paper, namely the Young measure and the Chacon's biting lemma. Actually, these two notions are very related as was observed in Ball and Murat [2].

Proposition 3.5. *(Young measures) If f^n is a sequence of functions bounded in $L^1(U;\mathbb{R}^m)$ where U is an open set of $\mathbb{R}^N$, then there exists a family $(\nu_x)_{x\in U}$ of probability measures on $\mathbb{R}^m$ (the Young measures), depending measurably on x and a subsequence also denoted f^n such that if $g:\mathbb{R}^m\to\mathbb{R}$ is continuous, if $A\subset U$ is measurable and*

$$g(f^n) \rightharpoonup z(x) \quad \text{weakly in}\, L^1(A;\mathbb{R}),$$

then $g(.)\in L^1(\mathbb{R}^m;\nu_x)$ for a.e. $x\in A$ and

$$z(x) = \int_{\mathbb{R}^m} g(\lambda)d\nu_x(\lambda) \quad a.e. \quad x\in A.$$

In the case where f^n is bounded in $L^p(U;\mathbb{R}^m)$ for some $p>1$ (or when f^n is equi-integrable), we can always take $A=U$ and we have (extracting a subsequence)

$$g(f^n) \rightharpoonup \int_{\mathbb{R}^m} g(\lambda)d\nu_x(\lambda).$$

Proposition 3.6. *(Chacon limit) If f^n is a sequence of functions bounded in $L^1(U;\mathbb{R}^m)$ where U is an open set of $\mathbb{R}^N$, then there exists a function $f\in L^1(U;\mathbb{R}^m)$, a subsequence f^n and a non-increasing sequence of measurable sets E_k of U with $\lim_{k\to\infty}\mathcal{L}_N(E_k)=0$ (where $\mathcal{L}_N$ is the Lebesgue measure on $\mathbb{R}^N$) such that for all $k\in\mathbb{N}$, $f^n \rightharpoonup f$ weakly in $L^1(U-E_k;\mathbb{R}^m)$ as n goes to infinity. f is called the Chacon limit of f^n.*

It is easy to see that if f^n is equi-integrable then the Chacon limit of f^n is equal to the weak limit of f^n in the sense of distribution.

If we consider continuous functions $g_k:\mathbb{R}^m\to\mathbb{R}^m$, $k\in\mathbb{N}$ satisfying the conditions :

(a) $g_k(\lambda)\to\lambda$ when $k\to\infty$, for each $\lambda\in\mathbb{R}^m$,

(b) $|g_k(\lambda)|\le C(1+|\lambda|)$, for all $k\in\mathbb{N}$ and $\lambda\in\mathbb{R}^m$,

(c) $\lim_{|\lambda|\to\infty}|\lambda|^{-1}|g_k(\lambda)|=0$ for each k,

then, under the hypotheses of Proposition 3.5, for each fixed k, the sequence of functions $g_k(f^n)$ is equi-integrable and hence (extracting a subsequence) converges weakly in $L^1(U;\mathbb{R}^m)$, to some f_k. Applying a diagonal process, as k goes to infinity, the sequence f_k converges strongly to some f in $L^1(U;\mathbb{R}^m)$. The limit f is the Chacon's limit of the subsequence f^n and it is given by

$$f(x) = \int_{\mathbb{R}^m} \lambda d\nu_x(\lambda) \quad a.e. \quad x\in U.$$

This gives an other possible definition of Chacon's limit which is equivalent to the one given in Proposition 3.6. For the proof of these results we refer to [2].

4. A priori estimates

4.1. **Free energy.** The second equation of (1) can be written as

$$\partial_t\psi + u.\nabla\psi = \mathrm{div}_R\Big[-\nabla u\cdot R\psi\Big] + div_R\Big[\psi_\infty\nabla_R\frac{\psi}{\psi_\infty}\Big]. \tag{28}$$

We define $\rho(t,x)=\int_B\psi dR$. Integrating (28) in R, we get the transport of ρ, namely $\partial_t\rho + u.\nabla\rho = 0$.

Multiplying (28) by $\log\frac{\psi}{\rho\psi_\infty}$ and integrating in R and x, we get

$$\partial_t\int_\Omega\int_B \psi\log(\frac{\psi}{\rho\psi_\infty}) - \psi + \rho\psi_\infty = \int_\Omega\int_B \nabla u\cdot R\,\nabla_R\mathcal{U}\psi - 4\int_\Omega\int_B\psi_\infty\left|\nabla_R\sqrt{\frac{\psi}{\psi_\infty}}\right|^2 dR \tag{29}$$

where we have used that $\nabla\psi_\infty = -\psi_\infty \nabla\mathcal{U}$.

The first equation of (1) yields the classical energy estimate for the Navier-Stokes equation

$$\partial_t \int_\Omega \frac{|u|^2}{2} = -\int_\Omega \nabla u : \tau - \nu \int_\Omega |\nabla u|^2. \tag{30}$$

Adding (29) and (30) yields the following decay of the free-energy

$$\partial_t \int_\Omega \int_B [\psi \log(\frac{\psi}{\rho\psi_\infty}) - \psi + \rho\psi_\infty] + \frac{|u|^2}{2} = -\nu \int_\Omega |\nabla u|^2 - 4\int_\Omega \int_B \psi_\infty \left|\nabla_R \sqrt{\frac{\psi}{\psi_\infty}}\right|^2. \tag{31}$$

Integrating in time, we get the following uniform bound for all $t > 0$

$$\int_\Omega \int_B [\psi \log(\frac{\psi}{\rho\psi_\infty}) - \psi + \rho\psi_\infty] + \frac{|u|^2}{2}\,(t) + \int_0^t \nu \int_\Omega |\nabla u|^2 + 4\int_\Omega \int_B \psi_\infty \left|\nabla_R \sqrt{\frac{\psi}{\psi_\infty}}\right|^2 = C_0. \tag{32}$$

To simplify the notations in the rest of this section, we will assume that $\rho_0(x) = 1$. The proof in the general case is identical and we will indicate the changes to be made at the end. The general idea is the following: When proving a priori estimates, one has just to replace ψ_∞ by $\rho(t,x)\psi_\infty$ and take advantage of the fact that ρ is just transported by the flow. When proving weak compactness, one can use that ρ^n converges strongly to ρ in all $L^p((0,T)\times\Omega)$ spaces and use $\rho^n(t,x)\psi_\infty$. Due to the local character of the proof of weak compactness, a simpler way is to just use ψ_∞ and so the calculations given in section 5 hold even when ρ_0 is not constant.

4.2. $\log^2$ **estimate.** The free energy only gives an $L\log L(\psi_\infty dR)$ bound on $\frac{\psi}{\psi_\infty}$. For some technical reasons, we will need to control a slightly higher growth of ψ in the R variable.

We introduce $\tilde\psi = \psi + a\psi_\infty$ for some $a > 1$. This is done to insure that $\log\frac{\tilde\psi}{\psi_\infty}$ does not take negative values. It will also add a new term in the equation which will not present any extra difficulties. Hence, $\tilde\psi$ solves

$$\partial_t\tilde\psi + u.\nabla\tilde\psi = \mathrm{div}_R\Big[-\nabla u\cdot R\tilde\psi\Big] + div_R\Big[\psi_\infty\nabla_R\frac{\tilde\psi}{\psi_\infty}\Big] - a\nabla u\cdot R\psi_\infty\nabla_R\mathcal{U}. \tag{33}$$

We first derive this extra bound in the case the domain Ω is bounded and then discuss the modification of the argument in the whole space case.

4.2.1. *Case of a bounded domain.* Multiplying (33) by $\log^2\frac{\tilde\psi}{\psi_\infty}$ and integrating by parts in R, we get

$$\begin{aligned}(\partial_t + u.\nabla_x)\int_B \tilde\psi[\log^2(\frac{\tilde\psi}{\psi_\infty}) - 2\log(\frac{\tilde\psi}{\psi_\infty}) + 2] &= -2ak\nabla_i u_j \int_B \frac{R_iR_j}{1-|R|^2}\psi_\infty \log^2\frac{\tilde\psi}{\psi_\infty}\\ &+ \int_B \nabla u\cdot R\tilde\psi\, 2\log(\frac{\tilde\psi}{\psi_\infty})\frac{\psi_\infty}{\tilde\psi}\nabla_R\frac{\tilde\psi}{\psi_\infty} - 8\int_B \psi_\infty\left|\nabla_R\sqrt{\frac{\tilde\psi}{\psi_\infty}}\right|^2\log(\frac{\tilde\psi}{\psi_\infty})\end{aligned} \tag{34}$$

The second term on the right hand side of (34) can be rewritten

$$2\int_B \nabla u\cdot R\psi_\infty\,\nabla_R\left(\frac{\tilde\psi}{\psi_\infty}\log(\frac{\tilde\psi}{\psi_\infty}) - \frac{\tilde\psi}{\psi_\infty}\right) = 2\int_B \nabla u\cdot R\,\nabla_R\mathcal{U}\tilde\psi\left(\log(\frac{\tilde\psi}{\psi_\infty}) - 1\right)$$

Taking the square root of (34), we get

$$(\partial_t + u.\nabla_x)\left(\int_B \tilde\psi[\log^2(\frac{\tilde\psi}{\psi_\infty}) - 2\log(\frac{\tilde\psi}{\psi_\infty}) + 2]\right)^{1/2} = \frac{-ak\nabla_i u_j \int_B \frac{R_iR_j}{1-|R|^2}\psi_\infty \log^2\frac{\tilde\psi}{\psi_\infty}}{\left(\int_B \tilde\psi[\log^2(\frac{\tilde\psi}{\psi_\infty}) - 2\log(\frac{\tilde\psi}{\psi_\infty}) + 2]\right)^{1/2}}$$

$$+\frac{\int_B \nabla u\cdot R\tilde\psi\, 2\log(\frac{\tilde\psi}{\psi_\infty})\frac{\psi_\infty}{\tilde\psi}\nabla_R\frac{\tilde\psi}{\psi_\infty}}{\left(\int_B \tilde\psi[\log^2(\frac{\tilde\psi}{\psi_\infty}) - 2\log(\frac{\tilde\psi}{\psi_\infty}) + 2]\right)^{1/2}} - 4\frac{\int_B \psi_\infty\left|\nabla_R\sqrt{\frac{\tilde\psi}{\psi_\infty}}\right|^2\log(\frac{\tilde\psi}{\psi_\infty})}{\left(\int_B \tilde\psi[\log^2(\frac{\tilde\psi}{\psi_\infty}) - 2\log(\frac{\tilde\psi}{\psi_\infty}) + 2]\right)^{1/2}} \tag{35}$$

$$= I_1 + I_2 + I_3.$$

Let us introduce the notation

$$N_2 = \left(\int_B \tilde\psi[\log^2(\frac{\tilde\psi}{\psi_\infty}) - 2\log(\frac{\tilde\psi}{\psi_\infty}) + 2]\right)^{1/2}. \tag{36}$$

To bound I_1 we use that, $\psi_\infty \log^2\frac{\tilde\psi}{\psi_\infty} \le C\tilde\psi$. Hence, the numerator of I_1 is bounded by $C|\nabla u|\int \frac{\tilde\psi}{1-|R|^2}dR$ which is clearly in $L^1((0,T)\times\Omega\times B)$. Indeed, by using (8) and Corollary (3.3), we see that

$$(\int \frac{\psi}{1-|R|^2}dR) \le C\left(\int_B \psi_\infty\left|\nabla_R\sqrt{\frac{\psi}{\psi_\infty}}\right|^2 dR\right)^{1/2}. \tag{37}$$

To bound the second term on the right hand side of (35), we use that the numerator can be bounded by

$$\left|\int_B \nabla u\cdot R\tilde\psi\ \log(\frac{\tilde\psi}{\psi_\infty})\frac{\psi_\infty}{\tilde\psi}\nabla_R\frac{\tilde\psi}{\psi_\infty}\right| \le$$

$$\le\ C|\nabla u|\left(\int_B \psi_\infty|\log(\frac{\tilde\psi}{\psi_\infty})|\left|\nabla_R\sqrt{\frac{\tilde\psi}{\psi_\infty}}\right|^2\right)^{1/2}\left(\int_B \psi_\infty|\log(\frac{\tilde\psi}{\psi_\infty})|\frac{\tilde\psi}{\psi_\infty}\right)^{1/2} \tag{38}$$

$$\le\ C|\nabla u|^2\left(\int_B \tilde\psi|\log(\frac{\tilde\psi}{\psi_\infty})|\right) + \left(\int_B \psi_\infty|\log(\frac{\tilde\psi}{\psi_\infty})|\left|\nabla_R\sqrt{\frac{\tilde\psi}{\psi_\infty}}\right|^2\right) \tag{39}$$

$$\le\ C|\nabla u|^2(1+a)^{1/2}\left(\int_B \tilde\psi\log^2(\frac{\tilde\psi}{\psi_\infty})\right)^{1/2} + \left(\int_B \psi_\infty|\log(\frac{\tilde\psi}{\psi_\infty})|\left|\nabla_R\sqrt{\frac{\tilde\psi}{\psi_\infty}}\right|^2\right) \tag{40}$$

Dividing (38) by N_2, we deduce that

$$I_2 \le C|\nabla u|^2 - \frac{1}{4}I_3. \tag{41}$$

Integrating (35) in time and space and using the fact that I_3 has a sign, we deduce the following a priori bound

(42)

$$\int_\Omega \left(\int_B \tilde{\psi}[\log^2(\frac{\tilde{\psi}}{\psi_\infty}) - 2\log(\frac{\tilde{\psi}}{\psi_\infty}) + 2] \right)^{1/2} (t) + \int_0^T \int_\Omega \frac{\int_B \psi_\infty \left| \nabla_R \sqrt{\frac{\tilde{\psi}}{\psi_\infty}} \right|^2 \log(\frac{\tilde{\psi}}{\psi_\infty})}{\left(\int_B \tilde{\psi}[\log^2(\frac{\tilde{\psi}}{\psi_\infty}) - 2\log(\frac{\tilde{\psi}}{\psi_\infty}) + 2] \right)^{1/2}} \leq C_T$$

for $0 \leq t \leq T$, if the initial condition satisfies $\int_\Omega \left(\int_B \tilde{\psi}_0[\log^2(\frac{\tilde{\psi}_0}{\psi_\infty}) - 2\log(\frac{\tilde{\psi}_0}{\psi_\infty}) + 2] \right)^{1/2} \leq C_0$. Hence, we see that (35) can be written as

$$(\partial_t + u.\nabla)N_2 = F_2 \tag{43}$$

where F_2 is in $L^1((0,T) \times \Omega)$.

It turns out that passing to the limit in the bound (42) is not clear. Actually, one can find sequences of functions $\tilde{\psi}^n$ such that (42) holds and the weak limit does not satisfy (42). This is the reason, we prefer to write the second bound as

(44)

$$\int_\Omega \left(\int_B g^2 \log(g^2) \psi_\infty dR \right)^{1/2} dx + \int_0^T \int_\Omega \int_B \frac{\psi_\infty \left| \nabla_R g \right|^2 \; dR}{\left(\int_B \tilde{\psi}[\log^2(\frac{\tilde{\psi}}{\psi_\infty}) - 2\log(\frac{\tilde{\psi}}{\psi_\infty}) + 2] \right)^{1/2}} \leq C_T$$

where g is given by $g = \sqrt{\frac{\tilde{\psi}}{\psi_\infty}} \log^{1/2}(\frac{\tilde{\psi}}{\psi_\infty})$.

4.2.2. *Case of unbounded domain.* In the case $\Omega = \mathbb{R}^D$, we first take c_1 and c_2 the two constants such that the function $\phi(x) = x[\log^2 x - 2\log x + c_1] + c_2$ satisfies the fact that $\phi(1+a) = \phi'(1+a) = 0$. This is achieved by taking $c_1 = 2 - \log^2(1+a)$ and $c_2 = 2(1+a)[\log(1+a) - 1]$. Notice also that the function $\phi(x)$ is nonnegative for $x \geq a$ since a is taken big enough. It is clear that the extra bound (6) implies that

$$\int_\Omega \frac{\int_B \phi(\frac{\tilde{\psi}_0}{\psi_\infty}) dR}{1 + \left[\int_B \phi(\frac{\tilde{\psi}_0}{\psi_\infty}) dR \right]^{1/2}} dx \leq C_0 \tag{45}$$

and hence, we can perform the same calculations as (34) and (35) with $\int_B \tilde{\psi}[\log^2(\frac{\tilde{\psi}}{\psi_\infty}) - 2\log(\frac{\tilde{\psi}}{\psi_\infty}) + 2] dR$ replaced by $\int_B \phi(\frac{\tilde{\psi}}{\psi_\infty}) dR$ and with the function $s \to \sqrt{s}$ used to go from (34) to (35) replaced by $s \to \frac{s}{1+\sqrt{s}}$ which behaves like $\phi_1(s) = \min(\sqrt{s}, s)$. The rest of the proof is identical.

4.2.3. *Case ρ is not constant.* In the case ρ is not constant and we are in a bounded domain, we have to modify (34) slightly and multiply by $\log^2 \frac{\tilde{\psi}}{\rho \psi_\infty}$. In the case we are also in an unbounded domain, we have to replace $\int_B \tilde{\psi}[\log^2(\frac{\tilde{\psi}}{\psi_\infty}) - 2\log(\frac{\tilde{\psi}}{\psi_\infty}) + 2] dR$ by $\int_B \phi(\frac{(1+a)\tilde{\psi}}{(\rho+a)\psi_\infty}) dR$. The extra factor $\frac{1+a}{\rho+a}$ is used to insure that when $\tilde{\rho}$ is at microscopic equilibrium, namely $\tilde{\psi} = (\rho+a)\psi_\infty$, the integrand reduces to $\phi(1+a)$. The rest of the proof is identical and yields at the end the following bound instead of (42)

$$\int_\Omega \phi_1 \left(\int_B \phi(\frac{(1+a)\tilde{\psi}}{(\rho+a)\psi_\infty}) \right) (t) + \int_0^T \int_\Omega \frac{\int_B \psi_\infty \left| \nabla_R \sqrt{\frac{\tilde{\psi}}{\psi_\infty}} \right|^2 \log(\frac{(1+a)\tilde{\psi}}{(\rho+a)\psi_\infty})}{1 + \left(\int_B \phi(\frac{(1+a)\tilde{\psi}}{(\rho+a)\psi_\infty}) \right)^{1/2}} \leq C_T. \tag{46}$$

One can then deduce from (46) that (42) and (44) hold with the integration set Ω replaced by any compact K of $\mathbb{R}^D$.

5. Weak compactness

As it is classical when proving global existence of weak solutions, it is enough to prove the weak compactness of a sequence of weak solutions satisfying the a priori estimates of the previous section. In the next section, we present one way of approximating the system. We consider (u^n, ψ^n) a sequence of weak solutions to (1) satisfying, uniformly in n, the free energy bound (32) and the $\log^2$ bound (42) with an initial data (u_0^n, ψ_0^n) such that (u_0^n, ψ_0^n) converge strongly to (u_0, ψ_0) in $L^2(\Omega)\times L^1_{loc}(\Omega; L^1(B))$ and $\psi_0^n \log \frac{\rho_0^n \psi_0^n}{\psi_\infty} - \psi_0^n + \rho_0^n \psi_\infty$ converges strongly to $\psi_0 \log \frac{\psi_0}{\rho_0 \psi_\infty} - \psi_0 + \rho_0 \psi_\infty$ in $L^1(\Omega \times B)$. We also assume that (u^n, ψ^n) has some extra regularity with bounds that depend on n such that we can perform all the following calculations.

We extract a subsequence such that u^n converges weakly to u in $L^p((0,T); L^2(\Omega)) \cap L^2((0,T); H^1_0(\Omega))$ and ψ^n converges weakly to ψ in $L^p((0,T); L^1_{loc}(\Omega \times B))$ for each $p < \infty$. We would like to prove that (u, ψ) is still a solution of (1). The main difficulty is to pass to the limit in the nonlinear term $\nabla u^n R \psi^n$ appearing in the second equation of (1).

We introduce $g^n = \sqrt{\frac{\tilde\psi^n}{\psi_\infty}} \log^{1/2}(\frac{\tilde\psi^n}{\psi_\infty})$ and $f^n = \sqrt{\frac{\tilde\psi^n}{\psi_\infty}}$ where $\tilde\psi^n = \psi^n + a\psi_\infty$ and $a > 1$ is any constant. We also assume, extracting a subsequence if necessary, that g^n and f^n converge weakly to some g and f in $L^p((0,T); L^2_{loc}(\Omega \times B,\, dx\psi_\infty dR))$ for each $p < \infty$. To prove that (u, ψ) is a solution of (1), it will be enough to prove that $(g^n)^2 = \frac{\tilde\psi^n}{\psi_\infty} \log(\frac{\tilde\psi^n}{\psi_\infty})$ converges weakly to $g^2 = \frac{\tilde\psi}{\psi_\infty} \log(\frac{\tilde\psi}{\psi_\infty})$, namely that g^n converges strongly to g in $L^2((0,T); L^2(\Omega\times B,\, dx\psi_\infty dR))$.

First, it is clear that $u, \tilde\psi$ and g satisfy the same a priori estimates that the sequence $u^n, \tilde\psi^n$ and g^n satisfy since all those functionals have good convexity properties. In particular it is clear that u, ψ satisfy (32). We just point out that to pass to the limit in the last term on the left hand side of (32), we can use the fact that the function $\phi_2(x,y) = \frac{x^2}{y}$ is convex. To pass to the limit in (44), we also use the fact that $\phi_2(x,y)$ is convex. Hence, we deduce that

$$\sup_{0\le t\le T} \int_\Omega \left(\left(\int_B g^2 \log(g^2) \psi_\infty dR \right)^{1/2} + \overline{N_2^n} \right) dx(t) + \int_0^T \int_\Omega \int_B \frac{\psi_\infty \left|\nabla_R g\right|^2}{\overline{N_2^n}} \le C_T \tag{47}$$

where $\overline{N_2^n}$ is the weak limit of $\left(\int_B \tilde\psi^n [\log^2(\frac{\tilde\psi^n}{\psi_\infty}) - 2\log(\frac{\tilde\psi^n}{\psi_\infty}) + 2] \right)^{1/2}$.

Dividing (33) by ψ_∞ we get

$$\partial_t \frac{\tilde\psi^n}{\psi_\infty} + u^n.\nabla \frac{\tilde\psi^n}{\psi_\infty} = \operatorname{div}_R\Big[-\nabla u^n \cdot R \frac{\tilde\psi^n}{\psi_\infty}\Big] + \nabla\mathcal{U}.\nabla u^n R \frac{\tilde\psi^n}{\psi_\infty} + \Delta_R \frac{\tilde\psi^n}{\psi_\infty} - \nabla\mathcal{U}.\nabla_R \frac{\tilde\psi^n}{\psi_\infty} - a\nabla u^n \cdot R.\nabla_R \mathcal{U}. \tag{48}$$

From (48), we deduce that for any smooth function Θ from $(0,\infty)$ to $\mathbb{R}$, we have

$$\begin{aligned} \partial_t \Theta(\frac{\tilde\psi^n}{\psi_\infty}) + u^n.\nabla\Theta(\frac{\tilde\psi^n}{\psi_\infty}) = &-\nabla u^n R \cdot \nabla_R \Theta(\frac{\tilde\psi^n}{\psi_\infty}) + \nabla_R \mathcal{U}.\nabla u R \frac{\tilde\psi^n}{\psi_\infty} \Theta'(\frac{\tilde\psi^n}{\psi_\infty}) \\ &+ \Delta\Theta(\frac{\tilde\psi^n}{\psi_\infty}) - \nabla_R\mathcal{U} \cdot \nabla_R \Theta(\frac{\tilde\psi^n}{\psi_\infty}) - \Theta''(\frac{\tilde\psi^n}{\psi_\infty}) |\nabla_R \frac{\tilde\psi^n}{\psi_\infty}|^2 \\ &- 2ak \nabla_i u^n_j \frac{R_i R_j}{1-|R|^2} \Theta'(\frac{\tilde\psi^n}{\psi_\infty}). \end{aligned} \tag{49}$$

We take $\Theta(t) = t^{1/2}\log^{1/2}(t)$ and recall that $g^n = \Theta(\frac{\tilde\psi^n}{\psi_\infty})$. We introduce the following defect measures $\gamma_{ij}, \gamma'_{ij}$ and β_{ij} such that

$$\nabla u^n g^n \to \nabla u g + \gamma, \qquad \nabla u^n \frac{\tilde\psi^n}{\psi_\infty}\Theta'(\frac{\tilde\psi^n}{\psi_\infty}) \to \nabla u \overline{\frac{\tilde\psi^n}{\psi_\infty}\Theta'(\frac{\tilde\psi^n}{\psi_\infty})} + \gamma' \tag{50}$$
$$\nabla u^n \tilde\psi^n \to \nabla u \tilde\psi + \beta$$

where $\gamma, \gamma' \in L^2((0,T)\times\Omega\times B)$ and $\beta \in L^2((0,T)\times\Omega; L^1(B))$ are matrix valued.

On one hand, passing to the limit in (49) with $\Theta(t) = t^{1/2}\log^{1/2}(t)$, we get

$$\begin{aligned}\partial_t g + u.\nabla g = {} & \mathrm{div}_R\Big[-\nabla_i u_j R_j g - \gamma_{ij}R_j\Big] \\ & + \nabla_R\mathcal{U}.\nabla u R\overline{\frac{\tilde\psi^n}{\psi_\infty}\Theta'(\frac{\tilde\psi^n}{\psi_\infty})} + \nabla_R\mathcal{U}R : \gamma' \\ & + \frac{1}{\psi_\infty}\mathrm{div}_R\Big[\psi_\infty\nabla_R g\Big] + \overline{\frac{|\nabla_R f^n|^2(\log^{1/2}+\log^{-3/2})(\frac{\tilde\psi^n}{\psi_\infty})}{f^n}} \\ & - ak\,\overline{\Theta'(\frac{\tilde\psi^n}{\psi_\infty})\nabla_i u_j^n}\frac{2R_iR_j}{1-|R|^2}\end{aligned} \tag{51}$$

where, here and below, $\overline{F_n}$ denotes the weak limit of F_n and where we have used that

$$\begin{cases}\Theta'(s) = \dfrac{1}{2}s^{-1/2}(\log^{1/2}(s) + \log^{-1/2}(s)) \\ \Theta''(s) = -\dfrac{1}{4}s^{-3/2}(\log^{1/2}(s) + \log^{-3/2}(s)).\end{cases} \tag{52}$$

Multiplying by g, we get

$$\begin{aligned}\partial_t g^2 + u.\nabla g^2 = {} & \mathrm{div}_R\Big[-\nabla u R g^2\Big] + \nabla u R\cdot\nabla_R\mathcal{U}\overline{\Big(\frac{\tilde\psi^n}{\psi_\infty}\Theta'(\frac{\tilde\psi^n}{\psi_\infty})\Big)}2g \\ & - \mathrm{div}_R(\gamma_{ij}R_j)2g + \nabla_R\mathcal{U}R : \gamma' 2g \\ & + \frac{1}{\psi_\infty}\mathrm{div}_R\Big[\psi_\infty\nabla_R g^2\Big] - 2|\nabla_R g|^2 \\ & + \overline{\frac{|\nabla_R f^n|^2(\log^{1/2}+\log^{-3/2})(\frac{\tilde\psi^n}{\psi_\infty})|^2}{f^n}}2g - ak\,\overline{\Theta'(\frac{\tilde\psi^n}{\psi_\infty})\nabla_i u_j^n}\frac{4R_iR_j}{1-|R|^2}g\end{aligned} \tag{53}$$

Multiplying (53) by ψ_∞ and integrating in R yields

$$\begin{aligned}(\partial_t + u.\nabla)\int_B \psi_\infty g^2 = {} & -\nabla u : \tau\left(\psi_\infty\Big(g^2 - 2g\overline{\frac{\tilde\psi^n}{\psi_\infty}\Theta'(\frac{\tilde\psi^n}{\psi_\infty})}\Big)\right) \\ & - \int_B \mathrm{div}_R(\gamma_{ij}R_j)2g\psi_\infty + \nabla_R\mathcal{U}R : \gamma' 2g\psi_\infty \\ & + \int_B \psi_\infty\overline{\frac{|\nabla_R f^n|^2(\log^{1/2}+\log^{-3/2})(\frac{\tilde\psi^n}{\psi_\infty})|^2}{f^n}}2g - 2\psi_\infty|\nabla_R g|^2 \\ & - \int_B \psi_\infty ak\,\overline{\Theta'(\frac{\tilde\psi^n}{\psi_\infty})\nabla_i u_j^n}\frac{4R_iR_j}{1-|R|^2}g\end{aligned} \tag{54}$$

where we recall that $\tau_{ij}(\psi) = 2k\int_B \psi \frac{R_iR_j}{1-|R|^2}dR$. Here, there is a small problem of definition: The terms on the second line of the right hand side are not well defined in the sense of distribution and we need some further analysis to make sense of them. Also, the transport term is not well defined even if we write it as $\mathrm{div}(u\int_B \psi_\infty g^2)$. Actually, as we will see later, we will not use (54) but a renormalized form of it. Indeed, we will construct in the next subsection a renormalizing factor N that satisfies $(\partial_t + u.\nabla)\frac{1}{N} = 0$ and we will make sense of (54) after dividing each term by N^4.

On the other hand, passing to the limit in the equation satisfied by $\tilde{\psi}_n$, we get

$$\partial_t\tilde{\psi} + u.\nabla\tilde{\psi} = \mathrm{div}_R\Big[-\nabla u\cdot R\tilde{\psi} - \beta_{ij}R_j\Big] + div_R\Big[\psi_\infty\nabla_R\frac{\tilde{\psi}}{\psi_\infty}\Big] - 2ak\nabla u : \psi_\infty\frac{R_iR_j}{1-|R|^2}. \tag{55}$$

Besides, $\tilde{\psi}^n\log(\frac{\tilde{\psi}^n}{\psi_\infty})$ satisfies

$$\begin{aligned}(\partial_t + u^n.\nabla)\left[\int_B \tilde{\psi}^n\log(\frac{\tilde{\psi}^n}{\psi_\infty})\right] &= \nabla u^n : \tau(\tilde{\psi}^n)\\ &- 4\int_B \psi_\infty\left|\nabla_R\sqrt{\frac{\tilde{\psi}^n}{\psi_\infty}}\right|^2 - 2ak\int_B \nabla u^n\frac{R_iR_j}{1-|R|^2}\psi_\infty\log(\frac{\tilde{\psi}^n}{\psi_\infty}).\end{aligned}\tag{56}$$

We would like to pass to the limit weakly in (56) and deduce that

$$\begin{aligned}(\partial_t + u.\nabla)\int_B \overline{\tilde{\psi}^n\log(\frac{\tilde{\psi}^n}{\psi_\infty})} &= \nabla u : \tau(\tilde{\psi}) + \int_B \beta_{ij}\frac{R_iR_j}{1-|R^2|}\\ &- \int_B \psi_\infty\overline{\left|\nabla_R\sqrt{\frac{\tilde{\psi}^n}{\psi_\infty}}\right|^2} - 2ak\int_B \overline{\log(\frac{\tilde{\psi}^n}{\psi_\infty})\nabla u^n}\frac{R_iR_j}{1-|R|^2}\psi_\infty.\end{aligned}\tag{57}$$

However, we can not use (50) to pass to the limit in $\nabla u^n : \tau(\tilde{\psi}^n) = \int_B \nabla u^n\frac{R_iR_j}{1-|R|^2}\tilde{\psi}^n$ and deduce that

$$\nabla u^n : \tau(\tilde{\psi}^n) \rightharpoonup \nabla u : \tau(\tilde{\psi}) + \int_B \beta_{ij}\frac{R_iR_j}{1-|R^2|} \tag{58}$$

since $\nabla u^n\frac{R_iR_j}{1-|R|^2}\tilde{\psi}^n$ is only bounded in $L^1(dt\,dx\,dR)$. Besides, we can not pass to the limit in the transport term even if we write it in divergence form.

To overcome these difficulties, we will divide (56) by $1+\delta N_2^n$ where N_2^n solves (43) before passing to the limit. Then, we will send δ to zero.

To be able to deal with the limit δ to zero, we need to renormalize (56) too. We denote $N_1^n = \int_B \tilde{\psi}^n\log(\frac{\tilde{\psi}^n}{\psi_\infty})$, $N_2^n = \left(\int_B \tilde{\psi}^n[\log^2(\frac{\tilde{\psi}^n}{\psi_\infty}) - 2\log(\frac{\tilde{\psi}^n}{\psi_\infty}) + 2]\right)^{1/2}$ and $\theta_\kappa(s) = \frac{s}{1+\kappa s}$. We first multiply (56) by $\theta'_\kappa(N_1^n)$ and get an equation for $\theta_\kappa(N_1^n)$. Dividing the resulting equation by $1+\delta N_2^n$, using (43) and taking the weak limit when n goes to infinity (extracting a subsequence if necessary), we get for $\kappa, \delta > 0$

$$
\begin{aligned}
(\partial_t + u.\nabla)\overline{\frac{\theta_\kappa(N_1^n)}{1+\delta N_2^n}} =& \overline{\nabla u^n : \frac{\tau(\tilde{\psi}^n)}{(1+\delta N_2^n)(1+\kappa N_1^n)^2}} - \overline{\frac{1}{(1+\delta N_2^n)(1+\kappa N_1^n)^2}\int_B \psi_\infty \left|\nabla_R \sqrt{\frac{\tilde{\psi}^n}{\psi_\infty}}\right|^2} \\
&- \overline{\frac{2ak}{(1+\delta N_2^n)(1+\kappa N_1^n)^2}\int_B \nabla u^n \frac{R_i R_j}{1-|R|^2}\psi_\infty \log(\frac{\tilde{\psi}^n}{\psi_\infty})} \\
&- \overline{\frac{\delta F^n}{(1+\delta N_2^n)^2}}\theta_\kappa(N_1^n).
\end{aligned} \tag{59}
$$

Now, we can send δ to zero. Notice that due to the fact that $\theta_\kappa(N_1^n)$ is bounded and that F^n is bounded in L^1, we deduce that the last term goes to zero when δ goes to zero. Then, we send κ to zero and recover at the limit

$$
\begin{aligned}
(\partial_t + u.\nabla)\theta =& \overline{\nabla u^n : \tau(\tilde{\psi}^n)}^{\delta,\kappa} - \overline{\int_B \psi_\infty \left|\nabla_R \sqrt{\frac{\tilde{\psi}^n}{\psi_\infty}}\right|^2}^{\delta,\kappa} \\
&- \overline{2ak\int_B \nabla u^n \frac{R_i R_j}{1-|R|^2}\psi_\infty \log(\frac{\tilde{\psi}^n}{\psi_\infty})}^{\delta,\kappa}.
\end{aligned} \tag{60}
$$

where $\theta = \lim_{\kappa\to 0}\lim_{\delta\to 0}\overline{\frac{\theta_\kappa(N_1^n)}{1+\delta N_2^n}} = \lim_{\kappa\to 0}\overline{\theta_\kappa(N_1^n)}$ is the Chacon limit of N_1^n and

$$
\overline{F_n}^{\delta,\kappa} = \lim_{\kappa\to 0}\lim_{\delta\to 0}\overline{\frac{F_n}{(1+\delta N_2^n)(1+\kappa N_1^n)^2}}
$$

for any sequence F_n bounded in L^1. We will prove in the next subsection that N_1^n is equiintegrable and hence that θ is also the weak limit of N_1^n. We also notice that if F_n is equi-integrable then $\overline{F_n}^{\delta,\kappa} = \overline{F_n}$.

To be really precise the term $u.\nabla\theta = \operatorname{div}(\theta u)$ on the left hand side of (60) is not well defined since θ is only in $L_t^\infty L_x^1$ and u is in $L_t^2\dot{H}_x^1$. To give sense to (60) we need to use a renormalizing factor. Actually, we made the same remark for the transport term in the equation (54). Recall that at the end, we would like to take the difference between (60) and (54) after dividing it by the factor N^4 that we are going to define now. The fact of dividing by N will insure that $\frac{\theta}{N}$ is bounded and that all the terms will make sense. So the point is to divide (59) by N^4 and then send δ and κ to zero.

5.1. **The renormalizing factor N.** We recall that from, (43), $\beta_M(N_2^n)$ solves

$$
(\partial_t + u^n.\nabla)\beta_M(N_2^n) = \beta_M'(N_2^n)F^n. \tag{61}
$$

where $\beta_M(s) = \theta_{1/M}(s) = \frac{sM}{M+s}$. Passing to the limit in (61), we get

$$
(\partial_t + u.\nabla)\overline{\beta_M(N_2^n)} = \overline{\beta_M'(N_2^n)F^n}. \tag{62}
$$

This equation does not seem to be very useful since the right hand side is a measure. To overcome this problem, we first introduce the unique a.e flow X^n in the sense of DiPerna and Lions [17] of u^n, solution of

$$
\partial_t X^n(t,x) = u^n(t, X(t,x)) \qquad X^n(t=0,x) = x. \tag{63}
$$

We also denote by X the a.e flow of u.

Let Q^n be the solution of (43) with F^n replaced by $|F^n|$ and taking the same initial data as N_2^n at $t = 0$. Hence,

$$\frac{d[\beta_M(Q^n)(t, X^n(t,x))]}{dt} = \beta_M'(Q^n)|F^n(t, X^n(t,x))| \tag{64}$$

where the equation holds in the sense of distribution.

Passing to the limit weakly in (64), we get

$$\frac{d\overline{[\beta_M(Q^n)(t, X^n(t,x))]}}{dt} = \overline{\beta_M'(Q^n)|F^n(t, X^n(t,x))|} \tag{65}$$

From the stability of the notion of a.e flow with respect to the weak limit of u^n to u, we know that $X^n(t,x)$ converges to $X(t,x)$ in L^1_{loc} and also that $(X^n(t)^{-1})(x)$ converges to $(X(t)^{-1})(x)$ in L^1_{loc}. This allows us to get the following equality of the weak limits

$$\overline{[\beta_M(Q^n)(t, X^n(t,x))]} = \overline{[\beta_M(Q^n)(t, X(t,x))]}. \tag{66}$$

Now, sending M to infinity in (65), we deduce that

$$\frac{d[Q(t, X(t,x))]}{dt} = F \tag{67}$$

where $Q = \lim_{M\to\infty} \overline{[\beta_M(Q^n)]}$ is the Chacon limit of Q^n and $F = \lim_{M\to\infty} \overline{\beta_M'(Q^n)|F^n(t, X^n(t,x))|}$. It is easy to see that $Q \in L^\infty(0,T; L^1(\Omega))$ and that $F \in \mathcal{M}((0,T)\times\Omega)$. Integrating in t, we deduce that a.e in $x \in \Omega$, we have

$$Q(t, X(t,x)) = Q(0,x) + \int_0^t F(s, X(s,x))\, ds \tag{68}$$

for a.e $t \in (0,T)$. Due to the fact that F is nonnegative, we deduce that $Q(t, X(t,x))$ is increasing in time. We define the normalizing factor N by the following

$$N(t, X(t,x)) = Q(T_0, X(T_0,x)) = Q(0,x) + \int_0^{T_0} F(s, X(s,x))\, ds \tag{69}$$

for $t \in (0, T_0)$ where $T_0 < T$ is a fixed time. In the sequel, we will denote $T = T_0$ and will not make the distinction between these two times. Notice that N is constant along the characteristics of u, that N is in $L^\infty(0,T; L^1(\Omega))$ and that $N(t, X(t,x))$ is in $L^1(\Omega; L^\infty(0,T))$. Moreover it is bounded from below by 1. Hence, it solves

$$(\partial_t + u.\nabla)\frac{1}{N} = 0.$$

Also, the following two inequalities hold

$$\overline{\beta_M(N_2^n)} \le \overline{\beta_M(Q^n)} \le Q \le N \tag{70}$$

and hence the weak limit of N_2^n which is equal to the Chacon limit of N_2^n is bounded by N. The fact that the weak limit of N_2^n is equal to its Chacon limit comes from the fact that the sequence N_2^n is equiintegrable. This is a simple consequence of the dissipation of the free energy and the weighted Sobolev inequality (24). Indeed, from (24), we deduce that $\sqrt{\frac{\psi^n}{\psi_\infty}}$ is bounded in $L^2((0,T)\times\Omega; L^p(\psi_\infty dR))$ on the other hand from the conservation of mass, we know that $\sqrt{\frac{\psi^n}{\psi_\infty}}$ is bounded in $L^\infty((0,T)\times\Omega; L^2(\psi_\infty dR))$. Interpolating between these two bounds, we easily deduce that $\sqrt{\frac{\psi^n}{\psi_\infty}}$ is bounded in $L^r((0,T)\times\Omega\times B, dtdx\psi_\infty dR)$ for some

$r > 2$ and hence N_2^n is equiintegrable. We also get that N_1^n is equiintegrable and hence θ which is the Chacon limit of N_1^n is equal to the weak limit of N_1^n.

5.2. **The term $\overline{\nabla u^n : \tau(\tilde{\psi}^n)}^{\delta,\kappa}$.** In this subsection, we will prove that $\overline{\nabla u^n : \tau(\tilde{\psi}^n)}^{\delta,\kappa} = \nabla u : \tau + \int_B \beta_{ij} \frac{R_i R_j}{1-|R^2|}$. This will follow from the following two lemmas

Lemma 5.1.

$$\overline{\frac{\nabla u^n : \tau(\tilde{\psi}^n)}{(1+\delta N_2^n)(1+\kappa N_1^n)^2}} = \int_B z^{\delta,\kappa} \frac{R_i R_j}{1-|R^2|} \tag{71}$$

where $z^{\delta,\kappa} = \overline{\frac{\tilde{\psi}^n \, \nabla u^n}{(1+\delta N_2^n)(1+\kappa N_1^n)^2}}$

Lemma 5.2. $z^{\delta,\kappa}$ *converges strongly to* $\overline{\nabla u^n \tilde{\psi}^n} = \nabla u \tilde{\psi} + \beta$ *in* $L^1((0,T)\times\Omega\times B; dt\, dx\, \frac{dR}{1-|R|})$ *when* δ *goes to zero and then* κ *goes to zero.*

Denoting $\tau^{n,\delta,\kappa} = \frac{\tau(\tilde{\psi}^n)}{(1+\delta N_2^n)(1+\kappa N_1^n)^2}$, we get that

Corollary 5.3.

$$\overline{\nabla u^n : \tau(\tilde{\psi}^n)}^{\delta,\kappa} = \lim_{\kappa\to 0}\lim_{\delta\to 0} \overline{\nabla u^n : \tau^{n,\delta,\kappa}} = \nabla u : \tau(\psi) + \int_B \beta_{ij} \frac{R_i R_j}{1-|R^2|} dR. \tag{72}$$

Proof of Lemma 5.1. The proof of (71) follows from the fact that $z^{n,\delta,\kappa} = \frac{\nabla u^n \tilde{\psi}^n}{\psi_\infty (1+\delta N_2^n)(1+\kappa N_1^n)^2}$ is equi-integrable in $L^1((0,T)\times\Omega\times B; dt\, dx\, \frac{\psi_\infty dR}{1-|R|})$ for δ,κ fixed. Indeed, consider the real valued function $\Phi(x) = x\log(1+x)+1$. It is enough to prove that $\Phi(z^{n,\delta,\kappa}) = \Phi\left(\frac{\nabla u^n \tilde{\psi}^n}{\psi_\infty (1+\delta N_2^n)(1+\kappa N_1^n)^2}\right)$ is bounded in $X = L^1((0,T)\times\Omega\times B; dt\, dx\, \frac{\psi_\infty dR}{1-|R|})$. To simplify notation, we denote $N^n = (1+\delta N_2^n)(1+\kappa N_1^n)^2$. Hence, it is enough to bound

$$\frac{\nabla u^n}{N^n}\left[\frac{\tilde{\psi}^n}{\psi_\infty}\log\left(\frac{\tilde{\psi}^n}{\psi_\infty}\right) + \frac{\tilde{\psi}^n}{\psi_\infty}\log\left(\frac{\nabla u^n}{N^n}\right)\right] \tag{73}$$

in X (see definition above).

To bound the first term appearing in (73) we use the Hardy type inequality (8) to get that

$$\frac{\nabla u^n}{N^n}\int_B \tilde{\psi}^n \log\left(\frac{\tilde{\psi}^n}{\psi_\infty}\right)\frac{1}{1-|R|}\, dR \tag{74}$$

$$\lesssim \frac{\nabla u^n}{N^n}\left[\int_B \tilde{\psi}^n \log\left(\frac{\tilde{\psi}^n}{\psi_\infty}\right)\right]^{1/2}\left[\int_B \psi_\infty \left|\nabla\Big(\sqrt{\frac{\tilde{\psi}^n}{\psi_\infty}}\log^{1/2}\frac{\tilde{\psi}^n}{\psi_\infty}\Big)\right|^2\right]^{1/2} \tag{75}$$

$$\lesssim |\nabla u^n|^2 + \frac{1}{(N^n)^2}\left[\int_B \tilde{\psi}^n \log\left(\frac{\tilde{\psi}^n}{\psi_\infty}\right)\right]\left[\int_B \psi_\infty \left|\nabla\Big(\sqrt{\frac{\tilde{\psi}^n}{\psi_\infty}}\log^{1/2}\frac{\tilde{\psi}^n}{\psi_\infty}\Big)\right|^2\right] \tag{76}$$

and using the a priori bound (42), we see that the last term in uniformly bounded in $L^1((0,T)\times\Omega)$.

To bound the second term in (73), we first use the inequality $x\, y \le C(x^2\log^2(x) + \frac{y^2}{\log^2 y})$ for $x, y \ge 2$ and then apply Jensen inequality. Hence,

$$\frac{\nabla u^n}{N^n}\log\left(\frac{\nabla u^n}{N^n}\right)\int_B \frac{\tilde{\psi}^n}{\psi_\infty}\frac{\psi_\infty}{1-|R|}\,dR \tag{77}$$

$$\lesssim |\nabla u^n|^2 + \frac{1}{(N^n)^2}\left[\int_B \frac{\tilde{\psi}^n}{\psi_\infty}\frac{\psi_\infty}{1-|R|}\,dR\right]^2 \log^2\left[\int_B \frac{\tilde{\psi}^n}{\psi_\infty}\frac{\psi_\infty}{1-|R|}\,dR\right] \tag{78}$$

$$\lesssim |\nabla u^n|^2 + \frac{1}{(N^n)^2}\left[\int_B \frac{\tilde{\psi}^n}{\psi_\infty}\log\left(\frac{\tilde{\psi}^n}{\psi_\infty}\right)\frac{\psi_\infty}{1-|R|}\,dR\right]^2 \tag{79}$$

and the last term can be bounded as in (74). We notice here that the last inequality implies in particular that $|\tau^{n,\delta,\kappa}|^2$ is equi-integrable in L^1 for fixed δ and κ. This is actually a very important fact that will be used again later.

Proof of Lemma 5.2. To prove this lemma, we use dominated convergence and monotone convergence. Indeed, $|z^{n,\delta,\kappa}|$ is decreasing in δ,κ, namely for $0<\delta\le\delta'$ and $0<\kappa\le\kappa'$, we have

$$|z^{n,\delta',\kappa'}| \le |z^{n,\delta,\kappa}| \le |\nabla u^n(\tilde{\psi}^n)|. \tag{80}$$

Passing to the limit weakly in n, we deduce that

$$\overline{|z^{n,\delta',\kappa'}|} \le \overline{|z^{n,\delta,\kappa}|} \le \overline{|\nabla u^n(\tilde{\psi}^n)|} \tag{81}$$

and by monotone convergence, we deduce that $G=\overline{\overline{|z^{n,\delta,\kappa}|}}^{\delta,\kappa}\in X$ and that for all $0<\delta$ and $0<\kappa$, we have $|z^{\delta,\kappa}|\le G$. Moreover, we have

$$|z^{\delta,\kappa}-z^{\delta',\kappa'}| \le \left|\overline{|z^{n,\delta,\kappa}|}-\overline{|z^{n,\delta',\kappa'}|}\right|. \tag{82}$$

Hence, there exists $g\in X$ such that $z^{\delta,\kappa}$ converges strongly to g in X. Now, we would like to prove that the limit g is equal to $\overline{\nabla u^n\tilde{\psi}^n}$. This follows from the fact that $\nabla u^n\tilde{\psi}^n$ is equi-integrable in $L^1((0,T)\times\Omega\times B;dt\,dx\,dR)$ (without the weight). Indeed, denoting $\Phi(x)=|x|\log^{1/2}(1+|x|)$, we have

$$\Phi(|\nabla u^n\tilde{\psi}^n|) \lesssim |\nabla u^n\tilde{\psi}^n|(\log(1+\tilde{\psi}^n)+\log(1+|\nabla u^n|)) \tag{83}$$

$$\lesssim \tilde{\psi}^n[|\nabla u^n|^2+\log(\tilde{\psi}^n)] \tag{84}$$

which is clearly bounded in $L^1(dt\,dx\,dR)$.

5.3. **Identification of $\int_B \beta_{ij}\frac{R_iR_j}{1-|R^2|}dR$.** In this subsection, we give a relation between β and some defect measure related to the lack of strong convergence of ∇u^n in L^2. To state the main proposition of this subsection, we introduce few notations. Let $u^n=v^n+w^n$ where v^n and w^nsolve

$$\begin{cases} \partial_t v^n-\Delta v^n+\nabla p_1^n=\nabla.\tau^n \\ v^n(t=0)=0 \end{cases} \tag{85}$$

$$\begin{cases} \partial_t w^n-\Delta w^n+\nabla p_2^n=-u^n.\nabla u^n \\ v^n(t=0)=u^n(t=0). \end{cases} \tag{86}$$

We further split w^n into $w_1^n+w_2^n$ where w_1^n is the solution with zero initial data and w_2^n is the solution with zero right hand side.

In the rest of this subsection we will use $\boldsymbol{\delta}$ to denote δ, κ. We define $v^{n,\boldsymbol{\delta}} = v^{n,\delta,\kappa}$ the solution of

$$\left\{ \begin{array}{l} \partial_t v^{n,\boldsymbol{\delta}} - \Delta v^{n,\boldsymbol{\delta}} + \nabla p_1^{n,\boldsymbol{\delta}} = \nabla . \tau^{n,\boldsymbol{\delta}} \\ v^{n,\boldsymbol{\delta}}(t=0) = 0 \end{array} \right. \tag{87}$$

Extracting a subsequence, we assume that $(\tau^{n,\boldsymbol{\delta}}, \nabla v^{n,\boldsymbol{\delta}}, \nabla v^n, \nabla w^n)$ converges weakly in L^2 to some $(\tau^{\boldsymbol{\delta}}, \nabla v^{\boldsymbol{\delta}}, \nabla v, \nabla w)$ and that

$$\overline{|\nabla v^{n,\boldsymbol{\delta}}|^2} = |\nabla v^{\boldsymbol{\delta}}|^2 + \mu^{\boldsymbol{\delta}} \tag{88}$$

for some defect measure $\mu^{\boldsymbol{\delta}} \in \mathcal{M}((0,T) \times \Omega)$. We also denote μ the limit of $\mu^{\boldsymbol{\delta}}$ when δ and then κ go to zero (extracting a subsequence), namely

$$\mu = \lim_{\kappa \to 0} \lim_{\delta \to 0} \mu^{\boldsymbol{\delta}} = \lim_{\boldsymbol{\delta} \to 0} \mu^{\boldsymbol{\delta}}. \tag{89}$$

Proposition 5.4. *We have*

$$\mu = - \int_B \beta_{ij} \frac{R_i R_j}{1 - |R^2|} dR. \tag{90}$$

Proof of Proposition 5.4.

We introduce the following weak limits

$$\overline{\tau^{n,\boldsymbol{\delta}} : \nabla v^{n,\boldsymbol{\delta}}} = W^{\boldsymbol{\delta}\boldsymbol{\delta}} \tag{91}$$

$$\overline{\tau^{n,\boldsymbol{\delta}} : \nabla v^n} = W^{\boldsymbol{\delta}}. \tag{92}$$

Step 1: First, we would like to prove that $W^{\boldsymbol{\delta}\boldsymbol{\delta}}$ and $W^{\boldsymbol{\delta}}$ have the same limit W when $\boldsymbol{\delta}$ goes to zero and that this limit is in L^1. To prove this, we introduce for $M > 0$, the following weak limits

$$\overline{\tau^{n,\boldsymbol{\delta}} 1_{|\tau^{n,\boldsymbol{\delta}}| \le M} : \nabla v^{n,\boldsymbol{\delta}}} = W_M^{\boldsymbol{\delta}\boldsymbol{\delta}} \tag{93}$$

$$\overline{\tau^{n,\boldsymbol{\delta}} 1_{|\tau^{n,\boldsymbol{\delta}}| > M} : \nabla v^{n,\boldsymbol{\delta}}} = W^{\boldsymbol{\delta}\boldsymbol{\delta}} - W_M^{\boldsymbol{\delta}\boldsymbol{\delta}} \tag{94}$$

$$\overline{\tau^{n,\boldsymbol{\delta}} 1_{|\tau^{n,\boldsymbol{\delta}}| \le M} : \nabla v^n} = W_M^{\boldsymbol{\delta}} \tag{95}$$

and

$$\overline{|\tau^{n,\boldsymbol{\delta}} 1_{|\tau^{n,\boldsymbol{\delta}}| \le M}|^2} = G_M^{\boldsymbol{\delta}} \qquad \overline{|\tau^{n,\boldsymbol{\delta}}|^2} = G^{\boldsymbol{\delta}}. \tag{96}$$

Since for a fixed $\boldsymbol{\delta}$, $|\tau^{n,\boldsymbol{\delta}}|^2$ is equi-integrable, we deduce that $G_M^{\boldsymbol{\delta}}$ converges to $G^{\boldsymbol{\delta}}$ in L^1 when M goes to infinity and is monotone in M. Also, by monotone convergence, we deduce that there exists $G \in L^1$ such that $G^{\boldsymbol{\delta}}$ converges to G in L^1 when $\boldsymbol{\delta}$ goes to zero. Actually, G is the weak limit of $|\tau^n|^2$ in the sense of Chacon.

Let us fix $\varepsilon > 0$. We choose $\boldsymbol{\delta}_0$ and M_0 such that for $\boldsymbol{\delta} < \boldsymbol{\delta}_0$ and $M > M_0$, we have $\|G - G^{\boldsymbol{\delta}}\|_{L^1} + \|G - G_M^{\boldsymbol{\delta}}\|_{L^1} \le \varepsilon$. We have

$$\overline{|\tau^{n,\boldsymbol{\delta}}|^2} = \overline{|\tau^{n,\boldsymbol{\delta}} 1_{|\tau^{n,\boldsymbol{\delta}}| \le M}|^2} + \overline{|\tau^{n,\boldsymbol{\delta}} 1_{|\tau^{n,\boldsymbol{\delta}}| > M}|^2} \tag{97}$$

$$= G_M^{\boldsymbol{\delta}} + \quad (G^{\boldsymbol{\delta}} - G_M^{\boldsymbol{\delta}}). \tag{98}$$

Hence, we deduce that for $\boldsymbol{\delta} < \boldsymbol{\delta}_0$ and $M > M_0$, we have for all n, $\|\tau^{n,\boldsymbol{\delta}} 1_{|\tau^{n,\boldsymbol{\delta}}| > M}\|_{L^2}^2 \le \varepsilon$ and hence, by Cauchy-Schwarz we deduce that $\|W^{\boldsymbol{\delta}\boldsymbol{\delta}} - W_M^{\boldsymbol{\delta}\boldsymbol{\delta}}\|_{L^1} \le C\sqrt{\varepsilon}$ and that $\|W^{\boldsymbol{\delta}} - W_M^{\boldsymbol{\delta}}\|_{L^1} \le C\sqrt{\varepsilon}$. Hence to prove that $\lim_{\boldsymbol{\delta}} W^{\boldsymbol{\delta}\boldsymbol{\delta}} = \lim_{\boldsymbol{\delta}} W^{\boldsymbol{\delta}}$, it is enough to prove it for the M approximation, namely that

$$\lim_{\boldsymbol{\delta}} W_M^{\boldsymbol{\delta}\boldsymbol{\delta}} = \lim_{\boldsymbol{\delta}} W_M^{\boldsymbol{\delta}}. \tag{99}$$

To prove (99), we first notice that $\tau^{n,\boldsymbol{\delta}} - \tau^n$ goes to zero in L^p for $p < 2$ when $\boldsymbol{\delta}$ goes to zero uniformly in n. Then, by parabolic regularity of the Stokes system, we deduce that $\|\nabla v^{n,\boldsymbol{\delta}} - \nabla v^n\|_{L^p((0,T)\times\Omega)}$ goes to zero when $\boldsymbol{\delta}$ goes to zero uniformly in n for $p < 2$. Hence, (99) holds.

Step 2: In this second step, we will compare the local energy identity of the weak limit of (87) with the weak limit of the local energy identity of (87).

On one hand, passing to the limit in (87) and multiplying by $v^{\boldsymbol{\delta}}$, we deduce that

$$\partial_t \frac{|v^{\boldsymbol{\delta}}|^2}{2} - \Delta \frac{|v^{\boldsymbol{\delta}}|^2}{2} + |\nabla v^{\boldsymbol{\delta}}|^2 + \operatorname{div}(p_1^{\boldsymbol{\delta}} v^{\boldsymbol{\delta}}) = \operatorname{div}(v^{\boldsymbol{\delta}}.\tau^{\boldsymbol{\delta}}) - \nabla v^{\boldsymbol{\delta}} : \tau^{\boldsymbol{\delta}} \tag{100}$$

On the other hand, reversing the order, we get

$$\partial_t \frac{|v^{\boldsymbol{\delta}}|^2}{2} - \Delta \frac{|v^{\boldsymbol{\delta}}|^2}{2} + |\nabla v^{\boldsymbol{\delta}}|^2 + \mu_{\boldsymbol{\delta}} + \operatorname{div}(p_1^{\boldsymbol{\delta}} v^{\boldsymbol{\delta}}) = \operatorname{div}(v^{\boldsymbol{\delta}}.\tau^{\boldsymbol{\delta}}) - W^{\boldsymbol{\delta}\boldsymbol{\delta}}. \tag{101}$$

For a justification of these two calculations, we refer to [48]. Comparing (100) and (101), we deduce that $W^{\boldsymbol{\delta}\boldsymbol{\delta}} = \nabla v^{\boldsymbol{\delta}} : \tau^{\boldsymbol{\delta}} - \mu_{\boldsymbol{\delta}}$. We would like now to send $\boldsymbol{\delta}$ to zero.

First, it is clear that $\tau^{\boldsymbol{\delta}}$ converges strongly to τ in L^2 when $\boldsymbol{\delta}$ goes to zero. Hence, $\nabla v^{\boldsymbol{\delta}}$ also converges to ∇v in L^2. Besides, from the energy estimate, we recall that u^n is bounded in $L^\infty((0,T); L^2(\Omega)) \cap L^2((0,T); \dot{H}^1(\Omega))$ and hence by Sobolev embeddings that u^n is bounded in $L^{\frac{2(D+2)}{D}}((0,T)\times\Omega)$ and that $u^n\nabla u^n$ is bounded in $L^{\frac{D+2}{D+1}}((0,T)\times\Omega)$. By parabolic regularity of the Stokes operator applied to (86) with zero initial data, we deduce that ∇w_1^n is bounded in $L^{\frac{D+2}{D+1}}((0,T); W^{1,\frac{D+2}{D+1}}\Omega)$ and that $\partial_t w_1^n$ is bounded in $L^{\frac{D+2}{D+1}}((0,T)\times\Omega)$. Since τ^n is bounded in L^2, we deduce from (85) that ∇v^n is also bounded in $L^2((0,T)\times\Omega)$ and hence ∇w^n is also bounded in $L^2((0,T)\times\Omega)$. Moreover, it is clear that ∇w_2^n is compact in $L^2((0,T)\times\Omega)$ and hence ∇w_1^n is also bounded in L^2 and from the previous bounds on ∇w_1^n, we deduce that ∇w_1^n is compact in $L^p((0,T)\times\Omega)$ for $p < 2$. Hence, we deduce that $\overline{\nabla w^n : \tau(\tilde{\psi^n})}^{\delta,\kappa} = \nabla w : \tau(\psi)$ (where we have used that $\tau^{n,\boldsymbol{\delta}}$ is equi-integrable for each fixed $\boldsymbol{\delta}$) and from Corollary 5.3 that $\lim_{\boldsymbol{\delta}} W^{\boldsymbol{\delta}\boldsymbol{\delta}} = \lim_{\boldsymbol{\delta}} W^{\boldsymbol{\delta}} = \overline{\nabla v^n : \tau(\tilde{\psi^n})}^{\delta,\kappa} = \nabla v : \tau(\psi) + \int_B \beta_{ij} \frac{R_i R_j}{1-|R^2|} dR$. Finally, we deduce that $\mu = \lim_{\boldsymbol{\delta}\to 0} \mu^{\boldsymbol{\delta}} = -\int_B \beta_{ij} \frac{R_i R_j}{1-|R^2|} dR$.

5.4. **Gronwall along the characteristics.** Taking the difference between (60) and (54) and dividing by N^4, we get (to be more precise, we have to take the difference between (59) and (54), divide by N^4 and then send $\boldsymbol{\delta}$ to zero):

$$
\begin{aligned}
&(\partial_t + u.\nabla)\frac{\eta}{N^4} \\
&\quad = \frac{1}{N^4}\left[\overline{\nabla u^n : \tau(\tilde\psi^n)}^{\delta,\kappa} + \nabla u : \tau\left(\psi_\infty\Big(g^2 - 2g\overline{\frac{\tilde\psi^n}{\psi_\infty}\Theta'(\frac{\tilde\psi^n}{\psi_\infty})}\Big)\right)\right] \\
&\qquad - \frac{1}{N^4}\int_B \psi_\infty\left[4\overline{|\nabla_R f^n|^2}^{\delta,\kappa} - 2|\nabla_R g|^2 + \overline{\frac{|\nabla_R f^n|^2(\log^{1/2}+\log^{-3/2})(\frac{\tilde\psi^n}{\psi_\infty})}{f^n}}2g\right] \\
&\qquad - \frac{2ak}{N^4}\int_B\left(\overline{\nabla u^n\Big(\log(\frac{\tilde\psi^n}{\psi_\infty})+1\Big)}^{\delta,\kappa} - \Big(\overline{2\Theta'(\frac{\tilde\psi^n}{\psi_\infty})\nabla_i u^n_j}\Big)g\right)\frac{R_iR_j}{1-|R|^2}\psi_\infty \\
&\qquad - \frac{2}{N^4}\int\psi_\infty\left[\gamma_{ij}R_j\nabla g - \nabla_R\mathcal{U}R : (\gamma-\gamma')g\right] \\
&\quad = -\sum_{i=1}^{4} A_i
\end{aligned}
\tag{102}
$$

where we denote the 4 terms appearing on the right hand side by $A_i, 1 \le i \le 4$ and we also denote $\eta = \overline{N_1^n}^{\delta,\kappa} - \int_B g^2\psi_\infty = \int_B[\overline{(g^n)^2} - g^2]\psi_\infty dR$. It measures the lack of strong convergence of g^n to g in $L^2(dtdx\psi_\infty dR)$. Notice that by the choice of the normalizing factor N, the defect measure $\frac{\eta}{N}$ is in L^∞.

First, we prove that A_2 is nonnegative, namely we have the following lemma

Lemma 5.5. *We have*

$$
A_2 \ge \frac{c}{N^4}\int_B \psi_\infty\overline{|\nabla_R(f^n - f)|^2}^{\delta,\kappa} = \frac{c}{N^4}\int_B\psi_\infty\varpi \tag{103}
$$

for some constant c.

For the proof, we rewrite $|\nabla_R g|^2$ as

$$
|\nabla_R g|^2 = \left|\overline{\nabla_R f^n(\log^{1/2}(f^n)^2 + \log^{-1/2}(f^n)^2)}\right|^2 \tag{104}
$$

$$
= \left|\overline{\nabla_R f^n(\log^{1/2}(f^n)^2)}\right|^2 + \left|\overline{\nabla_R f^n(\log^{-1/2}(f^n)^2)}\right|^2 \tag{105}
$$

$$
+ 2\overline{\nabla_R f^n(\log^{1/2}(f^n)^2)}\cdot\overline{\nabla_R f^n(\log^{-1/2}(f^n)^2)}. \tag{106}
$$

Hence, we deduce that

$$
A_2 = \frac{1}{N^4}\int_B\psi_\infty(\boldsymbol{\alpha}+\boldsymbol{\beta}+\boldsymbol{\gamma}) \tag{107}
$$

where $\boldsymbol{\alpha}, \boldsymbol{\beta}$ and $\boldsymbol{\gamma}$ are given by

$$
\frac{\boldsymbol{\alpha}}{2} = \overline{\frac{|\nabla_R f^n|^2(\log^{1/2}(\frac{\tilde\psi^n}{\psi_\infty}))}{f^n}}\,\overline{f^n\log^{1/2}(\frac{\tilde\psi^n}{\psi_\infty})} - \overline{(\nabla f^n)\log^{1/2}(\frac{\tilde\psi^n}{\psi_\infty})}^2 \tag{108}
$$

$$
\frac{\boldsymbol{\beta}}{2} = \overline{\frac{|\nabla_R f^n|^2(\log^{-3/2}(\frac{\tilde\psi^n}{\psi_\infty}))}{f^n}}\,\overline{f^n\log^{1/2}(\frac{\tilde\psi^n}{\psi_\infty})} - \overline{(\nabla f^n)\log^{-1/2}(\frac{\tilde\psi^n}{\psi_\infty})}^2 \tag{109}
$$

$$\frac{\gamma}{2} = 2\overline{|\nabla f^n|^2}^{\delta,\kappa} - 2\overline{(\nabla f^n)\log^{1/2}(\frac{\tilde{\psi}^n}{\psi_\infty})}\quad\overline{(\nabla f^n)\log^{-1/2}(\frac{\tilde{\psi}^n}{\psi_\infty})} \tag{110}$$

We introduce the Young measure $\nu_{t,x,R}(\Lambda,\lambda)$ associated to the sequence $(\nabla f^n, f^n)$ where $\Lambda \in \mathbb{R}^D$ and $\lambda \in \mathbb{R}$. Hence, the defect measure $\overline{|\nabla_R(f^n-f)|^2}^{\delta,\kappa}$ satisfies :

$$\overline{|\nabla_R(f^n-f)|^2} \geq \overline{|\nabla_R(f^n-f)|^2}^{\delta,\kappa} \geq \int |\Lambda - \int \Lambda' \nu_{t,x,R}(\Lambda',\lambda')|^2 \nu_{t,x,R}(\Lambda,\lambda) \tag{111}$$

$$= \frac{1}{2}\int\int |\Lambda-\Lambda'|^2 \nu_{t,x,R}(\Lambda',\lambda')\nu_{t,x,R}(\Lambda,\lambda) \tag{112}$$

Indeed, it is easy to see that $\overline{|\nabla_R(f^n-f)|^2}^{\delta,\kappa}$ is bounded from above by the weak limit and from below by the Chacon limit of $|\nabla_R(f^n-f)|^2$. In the sequel, we will drop the t, x and R dependence of ν and will denote $\nu' = \nu(\Lambda',\lambda')$ and $\nu = \nu(\Lambda,\lambda)$. Besides, $\boldsymbol{\alpha}, \boldsymbol{\beta}$ and $\boldsymbol{\gamma}$ satisfy

$$\boldsymbol{\alpha} \geq \int\int A(\Lambda,\lambda,\Lambda',\lambda')\nu(\Lambda',\lambda')\nu(\Lambda,\lambda) \tag{113}$$

and the same for $\boldsymbol{\beta}$ and $\boldsymbol{\gamma}$ with A replaced by B or C where A, B and C are given by

(114)
$$A = \frac{|\Lambda|^2\log^{1/2}(\lambda^2)}{\lambda}\lambda'\log^{1/2}(\lambda')^2 + \frac{|\Lambda'|^2\log^{1/2}(\lambda')^2}{\lambda'}\lambda\log^{1/2}(\lambda^2) - 2\Lambda.\Lambda'\log^{1/2}(\lambda^2)\log^{1/2}(\lambda')^2$$
(115)
$$B = \frac{|\Lambda|^2\log^{-3/2}(\lambda^2)}{\lambda}\lambda'\log^{1/2}(\lambda')^2 + \frac{|\Lambda'|^2\log^{-3/2}(\lambda')^2}{\lambda'}\lambda\log^{1/2}(\lambda^2) - 2\Lambda.\Lambda'\log^{-1/2}(\lambda^2)\log^{-1/2}(\lambda')^2$$
(116)
$$C = 2|\Lambda|^2 + 2|\Lambda'|^2 - 2\Lambda.\Lambda'\Big(\log^{1/2}(\lambda^2)\log^{-1/2}(\lambda'^2) + \log^{-1/2}(\lambda^2)\log^{1/2}(\lambda')^2\Big).$$

To prove lemma 5.5, it is enough to show that $A+B+C \geq \frac{c}{2}|\Lambda-\Lambda'|^2$. First, we rewrite $A+B+C$ as

$$A+B+C = |\Lambda|^2 B_1 + |\Lambda'|^2 B_2 - 2\Lambda.\Lambda' B_3 \tag{117}$$

$$= |\Lambda-\Lambda'|^2 + |\Lambda|^2(B_1-1) + |\Lambda'|^2(B_2-1) - 2\Lambda.\Lambda'(B_3-1) \tag{118}$$

where B_1, B_2 and B_3 are given by

$$B_1 = \frac{\log^{1/2}(\lambda^2)}{\lambda}\lambda'\log^{1/2}(\lambda')^2 + \frac{\lambda'\log^{1/2}(\lambda')^2}{\lambda\log^{3/2}(\lambda^2)} + 2 \tag{119}$$

$$B_2 = \frac{\log^{1/2}(\lambda'^2)}{\lambda'}\lambda\log^{1/2}(\lambda^2) + \frac{\lambda\log^{1/2}(\lambda^2)}{\lambda'\log^{3/2}(\lambda'^2)} + 2 \tag{120}$$

$$B_3 = \log^{1/2}(\lambda^2)\log^{1/2}(\lambda'^2) + \frac{1}{\log^{1/2}(\lambda^2)\log^{1/2}(\lambda'^2)} + \frac{\log^{1/2}(\lambda^2)}{\log^{1/2}(\lambda'^2)} + \frac{\log^{1/2}(\lambda'^2)}{\log^{1/2}(\lambda^2)} \tag{121}$$

Actually, we will prove that if a is chosen big enough then $(B_1-1)(B_2-1) \geq (B_3-1)^2$ from which we deduce that $A+B+C \geq |\Lambda-\Lambda'|^2$ and the lemma would follow. Indeed, after simple calculations, we get

$$
\begin{aligned}
&(B_1-1)(B_2-1)-(B_3-1)^2 = \\
&\quad \log^{1/2}(\lambda^2)\log^{1/2}(\lambda'^2)\left[\frac{\lambda}{\lambda'}+\frac{\lambda'}{\lambda}+2-2\frac{\log^{1/2}(\lambda^2)}{\log^{1/2}(\lambda'^2)}-2\frac{\log^{1/2}(\lambda'^2)}{\log^{1/2}(\lambda^2)}\right] \\
&\quad +2\left[\frac{\log^{1/2}(\lambda^2)}{\log^{1/2}(\lambda'^2)}+\frac{\log^{1/2}(\lambda'^2)}{\log^{1/2}(\lambda^2)}-2\right] \\
&\quad +\frac{1}{\log^{1/2}(\lambda^2)\log^{1/2}(\lambda'^2)}\left[\frac{\lambda\log(\lambda^2)}{\lambda'\log(\lambda'^2)}+\frac{\lambda'\log(\lambda'^2)}{\lambda\log(\lambda^2)}+2-2\frac{\log^{1/2}(\lambda^2)}{\log^{1/2}(\lambda'^2)}-2\frac{\log^{1/2}(\lambda'^2)}{\log^{1/2}(\lambda^2)}\right]
\end{aligned}
$$

We will prove that the three terms appearing inside the brackets are nonnegative. This is obvious for the second one since it is of the form $x+\frac{1}{x}-2$ for some $x>0$. We recall that since $(f^n)^2\geq a$, we get that $\lambda\geq\sqrt{a}$ on the support of ν. For the first bracket, we assume that $\lambda'\geq\lambda$ and write $\lambda'=\lambda(1+\varepsilon)$. Hence, the term in the first bracket is given by

$$1+\varepsilon+\frac{1}{1+\varepsilon}+2-2\sqrt{1+\frac{\log(1+\varepsilon)}{\log\lambda}}-2\frac{1}{\sqrt{1+\frac{\log(1+\varepsilon)}{\log\lambda}}} \tag{122}$$

and one can check easily that if $\lambda\geq\sqrt{a}$ is big enough then (122) is nonnegative. The same argument can be used for the third bracket. This end the proof of lemma 5.5.

To bound A_1, we first observe that

$$g^2-2g\overline{\frac{\tilde{\psi}^n}{\psi_\infty}\Theta'(\frac{\tilde{\psi}^n}{\psi_\infty})}=-\overline{f^n\log^{1/2}(f^n)^2}\;\overline{f^n\log^{-1/2}(f^n)^2} \tag{123}$$

and hence,

$$A_1=-\frac{1}{N^4}\left[\overline{\nabla u^n:\tau(\tilde{\psi}^n)}^{\delta,\kappa}+\nabla u:\tau\left(\psi_\infty\Big(g^2-2g\overline{\frac{\tilde{\psi}^n}{\psi_\infty}\Theta'(\frac{\tilde{\psi}^n}{\psi_\infty})}\Big)\right)\right] \tag{124}$$

$$=-\frac{1}{N^4}\left[\overline{\nabla u^n:\tau(\tilde{\psi}^n-\psi)}^{\delta,\kappa}+\nabla u:\tau(\psi-\psi_\infty(\overline{f^n\log^{1/2}(f^n)^2}\;\overline{f^n\log^{-1/2}(f^n)^2}))\right] \tag{125}$$

$$=\frac{1}{N^4}\left[\mu-\nabla u:\tau(\psi-\psi_\infty(\overline{f^n\log^{1/2}(f^n)^2}\;\overline{f^n\log^{-1/2}(f^n)^2}))\right] \tag{126}$$

By convexity, it is clear that $\overline{(f^n-f)^2}=\overline{(f^n)^2}-f^2\geq\overline{(f^n)^2}-\overline{f^n\log^{1/2}(f^n)^2}\;\overline{f^n\log^{-1/2}(f^n)^2}$ and hence,

$$|\tau(\psi-\psi_\infty(\overline{f^n\log^{1/2}(f^n)^2}\;\overline{f^n\log^{-1/2}(f^n)^2}))|\leq\left(\int_B\psi_\infty\overline{(f^n-f)^2}\int_B\psi_\infty\overline{|\nabla(f^n-f)|^2}^{\delta,\kappa}\right)^{1/2} \tag{127}$$

Hence,

$$-A_1\leq-\frac{\mu}{N^4}+C\frac{|\nabla u|}{N^4}\left(\int_B\psi_\infty\overline{(f^n-f)^2}\int_B\psi_\infty\overline{|\nabla(f^n-f)|^2}^{\delta,\kappa}\right)^{1/2} \tag{128}$$

$$\leq-\frac{\mu}{N^4}+C|\nabla u|^2\frac{\eta}{N^4}+\frac{1}{10N^4}\int_B\psi_\infty\overline{|\nabla(f^n-f)|^2}^{\delta,\kappa}. \tag{129}$$

The term between parentheses in the definition of A_3 can be written as

$$\overline{\frac{\nabla u^n}{f^n}\Big(\log^{1/2}(f^n)^2+\log^{-1/2}(f^n)^2\Big)\Big[f^n\log^{1/2}(f^n)^2-\overline{f^n\log^{1/2}(f^n)^2}\Big]} \tag{130}$$

If we denote $\nu_{t,x,R}(\Pi,\lambda)$ the Young measure associated to the sequence $(\nabla_x u^n, f^n)$, then we see easily that A_3 is given by

$$A_3=-\frac{2ak}{N^4}\int_B\int\int\Big(\frac{\Pi}{\lambda}(\log^{1/2}\lambda^2+\log^{-1/2}\lambda^2)-\frac{\Pi'}{\lambda'}(\log^{1/2}\lambda'^2+\log^{-1/2}\lambda'^2)\Big) \tag{131}$$

$$(\lambda\log^{1/2}\lambda^2-\lambda'\log^{1/2}\lambda'^2)\frac{R_iR_j}{1-|R|^2}\psi_\infty\,d\nu\,d\nu'\,dR. \tag{132}$$

The absolute value of the two factors inside the integral can be bounded respectively by

$$|\Pi-\Pi'|(\frac{\log^{1/2}\lambda^2}{\lambda}+\frac{\log^{1/2}\lambda'^2}{\lambda'})+(|\Pi|+|\Pi'|)(\frac{\log^{1/2}\lambda^2}{\lambda}-\frac{\log^{1/2}\lambda'^2}{\lambda'})\quad\text{and} \tag{133}$$

$$|\lambda-\lambda'|(\log^{1/2}\lambda^2+\log^{1/2}\lambda'^2).\quad\text{Hence} \tag{134}$$

$$|A_3|\le\frac{1}{10N^4}\int_B\int\int|\Pi-\Pi'|^2(\frac{\log\lambda^2}{\lambda}+\frac{\log\lambda'^2}{\lambda'})^2\frac{1}{1-|R|^2}\psi_\infty\,d\nu\,d\nu'\,dR \tag{135}$$

$$+\frac{C}{N^4}\int_B\int\int(|\Pi|+|\Pi'|)|\lambda-\lambda'|^2(\frac{\log\lambda^2}{\lambda^2}+\frac{\log\lambda'^2}{\lambda'^2})^2\frac{1}{1-|R|^2}\psi_\infty\,d\nu\,d\nu'\,dR \tag{136}$$

$$+\frac{C}{N^4}\int_B\int\int(1+|\Pi|+|\Pi'|)|\lambda-\lambda'|^2\frac{1}{1-|R|^2}\psi_\infty\,d\nu\,d\nu'\,dR \tag{137}$$

$$\le\frac{1}{10N^4}\mu+\frac{1}{10N^4}\kappa+\frac{C}{N^4}|\nabla u|^2\eta \tag{138}$$

Finally, to bound $-A_4$, we split it into two terms :

$$|A_4^1|\le\frac{2}{N^4}\int_B\psi_\infty|\gamma_{ij}||\nabla g|\ dR \tag{139}$$

$$\le\frac{1}{10N^4}\overline{|\nabla u^n-\nabla u|}^{\delta,\kappa}+\frac{C}{N^4}\overline{\Big(\int_B(g^n-g)|\nabla_R g|\psi_\infty\Big)^2} \tag{140}$$

$$\le\frac{1}{10N^4}\mu+\frac{C}{N^4}\int_B\psi_\infty|\nabla_R g|^2\int_B\psi_\infty\overline{(g^n-g)^2} \tag{141}$$

To bound A_4^2, we first consider the case $k>1$ where the term can be treated as A_4^1 using (7):

$$\begin{aligned}|A_4^2|&\le\frac{2}{N^4}\int_B\psi_\infty(|\gamma_{ij}|+|\gamma'_{ij}|)\frac{g}{1-|R|}\ dR\\ &\le\frac{1}{10N^4}\mu+\frac{C}{N^4}\int_B\psi_\infty\overline{(g^n-g)^2}\ dR\int_B\psi_\infty\frac{|g|^2}{(1-|R|)^2}\ dR\\ &\le\frac{1}{10N^4}\mu+\frac{C}{N^4}\Big(\int_B\psi_\infty|\nabla_R g|^2\Big)\eta.\end{aligned} \tag{142}$$

In the case $k\le1$, we have to use (10) instead of (7). Take $0\le\beta<k$ and $\gamma=\frac{1-\beta}{2}$

(143)
$$\left(\int_B \psi_\infty \frac{|g^n-g|\,g}{(1-|R|^2)}\,dR\right)^2 \quad \leq \int_B \psi_\infty \frac{(g^n-g)^2\log^{-\gamma}(f^n)^2}{(1-|R|^2)^{1-\beta}}\,dR \quad \int_B \psi_\infty \frac{g^2\log^{\gamma}(f^n)^2}{(1-|R|^2)^{1+\beta}}\,dR$$

To bound the second term, we use the following Young's inequality for $a, b \geq 1$ $ab \leq a\log^\gamma a + e^{(b^{\frac{1}{\gamma}})}$. We denote $d = 1-|R|^2$ and hence

$$\int_B \psi_\infty \frac{g^2\log^\gamma(f^n)^2}{(1-|R|^2)^{1+\beta}}\,dR \leq \int_B \psi_\infty\left[\frac{g^2}{d^{1+\beta}}\log^\gamma\frac{g^2}{d^{1+\beta}} + |f^n|^2\right]dR.$$

On the set $\{g^2 \geq \frac{1}{d^\varepsilon}\}$ where $\varepsilon = \frac{k-\beta}{2}$, we have $\log^\gamma \frac{g^2}{d^{1+\beta}} \leq C\log^\gamma g^2$. Besides, we have using (10)

$$\int_B \psi_\infty \frac{g^2\log^\gamma g^2}{d^{1+\beta}}dR \ \leq \left(\int_B \psi_\infty g^2\log g^2\right)^{\frac{1-\beta}{2}}\left(\int_B \psi_\infty(|\nabla_R g|^2+g^2)\right)^{\frac{1+\beta}{2}}.$$

On the set $\{g^2 \leq \frac{1}{d^\varepsilon}\}$, we have

$$\frac{g^2}{d^{1+\beta}}\log^\gamma\frac{g^2}{d^{1+\beta}} \leq \frac{C}{d^{1+\beta+\varepsilon}}\log^\gamma(\frac{1}{d})$$

which is integrable in the ball B with the measure $\psi_\infty dR$.

To bound the first term on the right hand side of (143), we first notice that

$$\overline{(g^n-g)^2\log^{-\gamma}(f^n)^2} \leq \overline{(f^n-f)^2\log^{1-\gamma}(C+(f^n-f)^2)}$$

which can be easily proved using Young measures. Besides, we have using (10)

$$\begin{aligned}
&|\int_B \psi_\infty \frac{(f^n-f)^2\log^{1-\gamma}(C+(f^n-f)^2)}{(1-|R|^2)^{1-\beta}}\,dR| \\
&\quad\leq \left(\int_B \psi_\infty(f^n-f)^2\log(C+(f^n-f)^2)\right)^{\frac{1+\beta}{2}}\left(\int_B \psi_\infty|\nabla_R(f^n-f)|^2\right)^{\frac{1-\beta}{2}} \\
&\quad\leq \frac{C}{\lambda^{\frac{2}{1+\beta}}}\left(\int_B \psi_\infty(f^n-f)^2\log(C+(f^n-f)^2)\right) + \lambda^{\frac{2}{1-\beta}}\left(\int_B \psi_\infty|\nabla_R(f^n-f)|^2\right).
\end{aligned}$$

for each $\lambda > 0$. Passing to the limit weakly (more precisely, applying $\overline{F_n}^{\delta,\kappa}$) to both sides and optimizing in λ, we deduce that,

(144)
$$\frac{1}{N^4}\overline{\left(\int_B \psi_\infty \frac{|g^n-g|\,g}{(1-|R|^2)}\,dR\right)^2} \leq \frac{C}{N^4}\left(\int_B \psi_\infty g^2\log g^2\right)^{\frac{1-\beta}{2}}\left(\int_B \psi_\infty(|\nabla_R g|^2+g^2)\right)^{\frac{1+\beta}{2}}\eta^{\frac{1+\beta}{2}}\varpi^{\frac{1-\beta}{2}}.$$

Putting all these estimates together, we deduce that

(145)
$$(\partial_t + u.\nabla)\frac{\eta}{N^4} + \frac{\mu+\varpi}{4N^4} \leq C|\nabla u|^2\frac{\eta}{N^4} + \frac{C}{N^4}\left(1+\int_B \psi_\infty|\nabla_R g|^2\right)\left(\int_B \psi_\infty g^2\log g^2\right)^{\frac{1-\beta}{1+\beta}}\eta.$$

We can take $\beta = 0$. Next, we observe that $\int_B \psi_\infty g^2\log g^2 dR \leq CN^2$. Indeed, if we introduce $h^n = g^n\log^{1/2} g^n$, we see that $N_2^n \geq \left(\int_B \psi_\infty(h^n)^2\right)^{1/2}$ and then it is easy to see

using that $(x,y) \to \frac{x^2}{y}$ is convex that

$$\overline{\left(\int_B \psi_\infty (h^n)^2\right)^{1/2}} \geq \left(\int_B \psi_\infty h^2\right)^{1/2}$$

from which we deduce the claim. Hence (145) becomes

$$\frac{d}{dt}\frac{\eta}{N^4}(t,X(t,x)) + \frac{\mu+\varpi}{4N^4}(t,X(t,x)) \leq C|\nabla u|^2 \frac{\eta}{N^4}(t,X(t,x)) + C\left[1+\int_B \psi_\infty \frac{|\nabla_R g|^2}{N}\right]\frac{\eta}{N}(t,X(t,x)). \tag{146}$$

First notice that the right hand side of (146) is in $L^1((0,T)\times K)$ for any bounded measurable set of Ω (To prove this, we can observe that $\frac{\eta}{N}$ is bounded and that using (47), the term between brackets in (146) is in $L^1((0,T)\times K)$). Hence (146) is well justified in the sense of distribution. In particular this justifies all the calculations done in this subsection starting from (102).

Now, since the term between brackets in (146) is in $L^1((0,T)\times K)$, for almost all x, $\int_0^T \left[1+\int_B \psi_\infty \frac{|\nabla_R g|^2}{N}\right](t,X(t,x))$ is finite. Besides, for almost all x, $N(t,X(t,x))$ (which is constant in t) is bounded. Hence, we deduce that for almost all x, $\int_0^T N^3\left[1+\int_B \psi_\infty \frac{|\nabla_R g|^2}{N}\right] + |\nabla u|^2(t,X(t,x))$ is finite. Hence, by Gronwall lemma, we deduce that for a.e x, we have for all $t<T$, $\frac{\eta(t,x)}{N^4} \leq \frac{\eta(0,x)}{N^4}e^{C_T(x)}$ and since $\eta(0,x)=0$ due to the initial strong convergence, we deduce that $\frac{\eta(t,x)}{N}^4 = 0$ and hence $\eta = 0$ and we deduce the strong convergence of g^n to g. This yields that (u,ψ) is a weak solution of (1) with the initial data (u_0,ψ_0).

6. Approximate system

In the previous section, we proved the weak compactness of a sequence of solutions to the system (1). Of course one has to construct a sequence of (approximate) weak solutions to which we can apply the strategy of the previous sections. The only thing we have to make sure is that the calculations done in the previous section can be made on the approximate system. We consider a sequence of global smooth solutions (u^n,ψ^n) to the following regularized system where k is some integer that depends on D. In particular one can take $k=1$ for $D=2$ or 3:

$$\begin{cases} \partial_t u^n + (u^n\cdot\nabla)u^n - \nu\Delta u^n + \frac{1}{n}(\Delta)^{2k}u^n + \nabla p^n = \mathrm{div}\tau^n, \quad \mathrm{div} u = 0, \\ \\ \partial_t\psi^n + u^n.\nabla\psi^n = \mathrm{div}_R\Big[-\nabla u^n\, R\psi^n + \beta\nabla\psi^n + \nabla\mathcal{U}\psi^n\Big] \\ \\ \tau_{ij}^n = \int_B (R_i\otimes\nabla_j\mathcal{U})\psi^n(t,x,R)dR \qquad (\nabla\mathcal{U}\psi^n + \beta\nabla\psi^n).n = 0 \text{ on } \partial B(0,R_0). \end{cases} \tag{147}$$

with a smooth initial condition (u_0^n,ψ_0^n) such that (u_0^n,ψ_0^n) converges strongly to (u_0,ψ_0) in $L^2(\Omega)\times L^1(\Omega\times B)$ and $\psi_0^n\log\frac{\psi_0^n}{\rho_0^n\psi_\infty} - \psi_0^n + \rho_0^n\psi_\infty$ converges strongly to $\psi_0\log\frac{\psi_0}{\rho_0\psi_\infty} - \psi_0 + \rho_0\psi_\infty$ in $L^1(\Omega\times B)$. We also assume that (6) holds uniformly in n. In the case Ω is a bounded domain of $\mathbb{R}^D$, we also impose the following boundary condition $u^n = \Delta u^n = ... = (\Delta)^{2k-1}u^n = 0$ at the boundary $\partial\Omega$.

We do not detail the proof of existence for the system (147). We only mention that we have to combine classical results about strong solutions to Navier-Stokes system with the study of the linear Fokker-Planck equation (see [51]). In particular the following operator was used

$$L\psi = -div(\psi_\infty\nabla\frac{\psi}{\psi_\infty}) \tag{148}$$

on the space $\mathcal{H} = L^2(\frac{dR}{\psi_\infty})$ and with domain

$$D(L) = \left\{ \psi \in \mathcal{H} | \psi_\infty \nabla \frac{\psi}{\psi_\infty} \in \mathcal{H}, \quad div(\psi_\infty \nabla \frac{\psi}{\psi_\infty}) \in \mathcal{H} \quad \text{and} \, \psi_\infty \nabla \frac{\psi}{\psi_\infty}|_{\partial B} = 0 \right\}. \tag{149}$$

Also the following two Hilbert spaces $\mathcal{H}^1$ and $\mathcal{H}^2$ are used in the construction :

$$\mathcal{H}^1 = \left\{ \psi \in \mathcal{H} \mid \quad \int \psi_\infty \left| \nabla \frac{\psi}{\psi_\infty} \right|^2 + \frac{\psi^2}{\psi_\infty} \, dR < \infty \right\} \tag{150}$$

$$\mathcal{H}^2 = \left\{ \psi \in \mathcal{H}^1 \mid \quad \int \left(div(\psi_\infty \nabla \frac{\psi}{\psi_\infty}) \right)^2 \frac{dR}{\psi_\infty} < \infty. \right\} \tag{151}$$

Following the the proof of existence given in [51], we can prove

Proposition 6.1. *Take $u_0^n \in H^s(\Omega)$ and $\psi_0^n \geq 0$ such that $\psi_0^n - \rho_0^n \psi_\infty \in H^s(\Omega; L^2(\frac{dR}{\psi_\infty}))$ with $\rho_0^n = \int \psi_0^n dR \in L^\infty(\Omega)$. Then, there exists a global unique solution (u^n, ψ^n) to (147) such that $(u^n, \psi^n - \rho^n \psi_\infty)$ is in $C([0,T); H^s) \times C([0,T); H^s(\mathbb{R}^N; L^2(\frac{dR}{\psi_\infty})))$ for all $0 < T$. Moreover, $u^n \in L^2([0,T); H^{s+k})$ and $\psi^n - \rho^n \psi_\infty \in L^2([0,T); H^s(\mathbb{R}^N; \mathcal{H}^1))$.*

Remark 6.2. *The proof is exactly the same as the proof of Theorem 2.1 of* [51] *with few differences:*

- *In* [51]*, we only had local existence whereas here, we have global existence since we have more regularity.*
- *Theorem 2.1 of* [51] *was stated in the whole space. Of course in the case of a bounded domain, we have to use energy bounds for Navier-Stokes written in a bounded domain.*
- *In theorem 2.1 of* [51] *we assumed that $\int \psi_0 dR = 1$. The result can be easily extended to this more general case. We also point out that there is a small mistake in the statement of the theorem 2.1 of* [51]*. Indeed, one has to read $\psi_0 - \psi_\infty \in H^s(\Omega; L^2(\frac{dR}{\psi_\infty}))$ instead of $\psi_0 \in H^s(\Omega; L^2(\frac{dR}{\psi_\infty}))$ when the problem is in the whole space.*

It is clear that the solutions constructed in Proposition 6.1 satisfy the free-energy bound (32) and the extra bound (42) (with Ω replaced by K in the whole space case).

Once we have our sequence of regular approximate solutions, we have to check that all the computations performed in the previous section can be done on this sequence (u^n, ψ^n). The only point to be checked is that Proposition 5.4 still holds since the rest of the proof only involves the transport equation. Now, v^n and w^nsolve

$$\begin{cases} \partial_t v^n - \Delta v^n + \frac{1}{n} \Delta^{2k} v^n + \nabla p_1^n = \nabla . \tau^n \\ v^n(t=0) = 0 \end{cases} \tag{152}$$

$$\begin{cases} \partial_t w^n - \Delta w^n + \frac{1}{n} \Delta^{2k} w^n + \nabla p_2^n = -u^n . \nabla u^n \\ v^n(t=0) = u^n(t=0) \end{cases} \tag{153}$$

and we define $v^{n,\boldsymbol{\delta}}$ the solution of

$$\begin{cases} \partial_t v^{n,\boldsymbol{\delta}} - \Delta v^{n,\boldsymbol{\delta}} + \frac{1}{n} \Delta^{2k} v^{n,\boldsymbol{\delta}} + \nabla p_1^{n,\boldsymbol{\delta}} = \nabla . \tau^{n,\boldsymbol{\delta}} \\ v^{n,\boldsymbol{\delta}}(t=0) = 0 \end{cases} \tag{154}$$

Step 1 of the proof of Proposition 5.4 is the same with the difference that one has to apply parabolic regularity for the perturbed Stokes operator which yields the same uniform in n estimate. Hence, we deduce that $\|\nabla v^{n,\boldsymbol{\delta}} - \nabla v^n\|_{L^p((0,T)\times\Omega)}$ goes to zero when $\boldsymbol{\delta}$ goes to zero uniformly in n for $p < 2$.

For the second step, we first notice that (100) remains the same since $\frac{1}{n}(\Delta)^{2k} u^n$ converges weakly to zero. Moreover, multiplying the first equation of (152) by $v^{n,\boldsymbol{\delta}}$, we get

$$\partial_t \frac{|v^{n,\delta}|^2}{2} - \Delta \frac{|v^{n,\delta}|^2}{2} + |\nabla v^{n,\delta}|^2 + G^{n,\delta} + \mathrm{div}(p_1^{n,\delta} v^{n,\delta}) = \mathrm{div}(v^{n,\delta}.\tau^{n,\delta}) - \tau^{n,\delta} : \nabla v^{n,\delta}. \tag{155}$$

where $G^{n,\delta}$ is given by

$$\begin{aligned} G^{n,\delta} = \tfrac{1}{n}[&\mathrm{div}_i(\nabla_i \Delta^{2k-1} v^{n,\delta}.v^{n,\delta} - \Delta^{2k-1} v^{n,\delta}.\nabla_i v^{n,\delta} + \\ &\nabla_i \Delta^{2k-2} v^{n,\delta}.\Delta v^{n,\delta} - ... - \Delta^k v^{n,\delta}.\nabla_i \Delta^{k-1} v^{n,\delta}) + \Delta^k v^{n,\delta}.\Delta^k v^{n,\delta}] \end{aligned} \tag{156}$$

Using the fact that $\frac{1}{n}\int_0^T\int_\Omega |\Delta^k v^{n,\delta}|^2$ and $\int_0^T\int_\Omega |\nabla v^{n,\delta}|^2$ are uniformly bounded, we deduce easily that $\overline{G^{n,\delta}} = \overline{|\Delta^k v^{n,\delta}|^2} \geq 0$ and hence passing to the limit in (157), we deduce that

$$\partial_t \frac{|v^{\delta}|^2}{2} - \Delta \frac{|v^{\delta}|^2}{2} + |\nabla v^{\delta}|^2 + \mu_\delta + \mathrm{div}(p_1^{\delta} v^{\delta}) \leq \mathrm{div}(v^{\delta}.\tau^{\delta}) - W^{\delta\delta}. \tag{157}$$

and hence Proposition 5.4 is replaced by an inequality $\mu \leq -\int_B \beta_{ij} \frac{R_i R_j}{1-|R^2|} dR$ which is the inequality that we need in the rest of the proof.

7. Conclusion

In this paper we gave a proof of existence of weak solutions to the system (1), using the fact that a sequence of regular solutions to the approximate system (147) converges weakly to a weak solution of (1). We would like here to mention few important open problems (with increasing level of difficulty, at least this is what the author thinks):

- *The zero diffusion limit in x.* If we add a diffusion term $\frac{1}{n}\Delta_x \psi$ in the Fokker-Planck equation of (1), then one can prove the global existence of weak solutions to the regularized model. A natural question is whether we recover a weak solution of the unregularized system (1) when n goes to zero. This is the object of a forthcoming paper [53]. The difficulty comes from the fact that the calculation of section 5 used in a critical way the fact that we had a transport equation in the x variable.
- *Relaxing the assumption* (42). This extra bound was only used to give some extra control on the stress tensor. Can we prove the same existence result without it ?
- *Other models.* A natural question is whether we can extend this to the Hooke model (where the system can be reduced to a macroscopic model). We were not able to perform this. The main difficulty is that we do not know whether the extra stress tensor τ is in L^2. Nevertheless, we know how to use the strategy to this paper to prove global existence for the FENE-P model [52].
- *Regularity in 2D.* Many works on polymeric flows are motivated by similar known results for the Navier-Stokes system. In particular a natural question is whether one can prove global existence of smooth solutions to (1) in 2D. We point out that this is known for the co-rotational model [47, 51]. Of course this seems to be a very difficult problem since, we only have an L^2 bound on τ and that an L^∞ bound on τ was necessary in the previously mentioned works. In particular the similar result is not known for the co-rotational Oldroyd-B model where one can prove L^p bounds on τ for each $p > 1$.
- *Is system* (1) *better behaved than Navier-Stokes.* One does not expect to prove results on (1) which are not known for Navier-Stokes since (1) is more complicated than Navier-Stokes. However, one can speculate that due to the polymers and the extra stress tensor, system (1) may behave better than Navier-Stokes and that one can prove global existence of smooth solutions to (1) even if such result is not proved or disproved for the Navier-Stokes system.

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